XIV. *On Mathematical Concepts of the Material World.*

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PREFACE.

THE object of this memoir is to initiate the mathematical investigation of various possible ways of conceiving the nature of the material world. In so far as its results are worked out in precise mathematical detail, the memoir is concerned with the possible relations to space of the ultimate entities which (in ordinary language) constitute the "stuff" in space. An abstract logical statement of this limited problem, in the form in which it is here conceived, is as follows: Given a set of entities which form the field of a certain polyadic (*i.e.*, many-termed) relation R, what "axioms" satisfied by R have as their consequence, that the theorems ot Euclidean geometry are the expression of certain properties of the field of R? If the set of entities are themselves to be the set of points of the Euclidean space, the problem, thus set, narrows itself down to the problem of the axioms of Euclidean geometry. The solution of this narrower problem of the axioms of geometry is assumed (*cf.* Part II., Concept I.) without proof in the form most convenient for this wider investigation. But in Concepts III., IV., and V., the entities forming the field of R are the "stuff," or part of the "stuff," constituting the moving material world. POINCARÉ* has used language which might imply the belief that, with the proper definitions, Euclidean geometry can be applied to express properties of the field of any polyadic relation whatever. His context, however, suggests that his thesis is, that in a certain sense (obvious to mathematicians) the Euclidean and certain other geometries are interchangeable, so that, if one can be applied, then each of the others can also be applied. Be that as it may, the problem, here discussed, is to find various formulations of axioms concerning R, from which, with appropriate definitions, the Euclidean geometry issues as expressing properties of the field of R. In view of the existence of change in the material world, the investigation has to be so conducted as to introduce, in its abstract form, the idea of time, and to provide for the definition of velocity and acceleration.

The general problem is here discussed purely for the sake of its logical (*i.e.*, mathematical) interest. It has an indirect bearing on philosophy by disentangling the essentials of the idea of a material world from the accidents of one particular concept. The problem might, in the future, have a direct bearing upon physical science if a concept widely different from the prevailing concept could be elaborated, which

* *Cf.* 'La Science et l'Hypothèse,' chap. III., at the end.

allowed of a simpler enunciation of physical laws. But in physical research so much depends upon a trained imaginative intuition, that it seems most unlikely that existing physicists would, in general, gain any advantage from deserting familiar habits of thought.

Part I. (i) consists of general considerations upon the nature of the problem and the method of procedure. Part I. (ii) contains a short explanation of the symbols used. Part II. is devoted to the consideration of three concepts, which embody the ordinary prevailing ideas upon the subject and slight variants from them. The present investigation has, as a matter of fact, grown out of the *Theory of Interpoints*, which is presented in Part III. (ii), and of the *Theory of Dimensions* of Part IV. (i). These contain two separate answers to the question: How can a point be defined in terms of lines? The well-known definition* of the projective point, as a bundle of lines, assumes the descriptive point. The problem is to define it without any such assumption. By the aid of these answers two concepts, IV. and V., differing very widely from the current concepts, have been elaborated. Concept V., in particular, appears to have great physical possibilities. Indeed, its chief difficulty is the bewildering variety of material which it yields for use in shaping explanations of physical laws. It requires, however, the discovery of some appropriate laws of motion before it can be applied to the ordinary service of physical science.

The Geometry throughout is taken to be three-dimensional and Euclidean. In Concept V. the definition of parallel lines and the "Euclidean" axiom receive new forms; also the "points at infinity" are found to have an intimate connection with the theory of the order of points on any straight line. The *Theory of Dimensions* is based on a new definition of the dimensions of a space.

The main object of the memoir is the development of the *Theory of Interpoints*, of the *Theory of Dimensions*, and of *Concept V.* The other parts are explanatory and preparatory to these, though it is hoped that they will be found to have some independent value.

PART I.—(i) GENERAL CONSIDERATIONS.

Definition.—The *Material World* is conceived as a set of relations and of entities which occur as forming the "fields" of these relations.

Definition.—The *Fundamental Relations* of the material world are those relations in it, which are not defined in terms of other entities, but are merely particularized by hypotheses that they satisfy certain propositions.

Definition.—The hypotheses, as to the propositions which the fundamental relations satisfy, are called the *Axioms* of that concept of the material world.

Definition.—Each complete set of axioms, together with the appropriate definitions and the resulting propositions, will be called a *Concept of the Material World.*

* Here in "Descriptive Geometry" straight lines are open, and three collinear points have a non-projective relation of order; in "Projective Geometry" straight lines are closed, and four collinear points have a projective relation of separation.

Definition.—The complete class of those entities, which are members of the fields of fundamental relations, is called the class of *Ultimate Existents.* This technical name is adopted without prejudice to any philosophic solution of the question of the true relation to existence of the material world as thus conceived.

Every concept of the material world must include the idea of time. Time must be composed of *Instants* (*cf.* BERTRAND RUSSELL, 'Mind,' N.S., vol. 10, No. 39). Thus *Instants of Time* will be found to be included among the ultimate existents of every concept.

Definition.—The class of ultimate existents, exclusive of the instants of time, will be called the class of *Objective Reals.*

The relation of a concept of the material world to some perceiving mind is not to be part of the concept. Also we have no concern with the philosophic problem of the relation of any, or all, of these concepts to existence.

In Geometry, as derived from the Greeks, the simple elements of space are *points*, and the science is the study of the relations between points. Points occur as members of the fields of these relations. Then matter (the ultimate "stuff" which occupies space) in its final analysis, even if it is continuous, consists of entities, here called *particles*, associated with the points by relations which are expressed by saying that a particle *occupies* (or *is at*) a point. Thus matter merely occurs as one portion of the field of this relation of occupation; the other portion consists of points of space and of instants of time. Thus "occupation" is a triadic relation holding in each specific instance between a particle of matter, a point of space, and an instant of time. According to this concept of a material world, which we will call the *Classical Concept*, the class of ultimate existents is composed of three mutually exclusive classes of entities, namely, *points of space*, *particles of matter*, and *instants of time.* Corresponding to these classes of entities there exist the sciences of *Geometry*, of *Chronology*, which may be defined as the theory of time considered as a one-dimensional series ordinally similar to the series of real numbers, and of *Dynamics.* There appears to be no science of matter apart from its relations to time and space.

Opposed to the classical concept stands LEIBNIZ's theory of the Relativity of Space. This is not itself a concept of the material world, according to the narrow definition here given. It is merely an indication of a possible type of concepts alternative to the classical concept. It is not very obvious how to state this theory in the precise nomenclature here adopted. The theory at least means that the points of space, as conceived in the classical concept, are not to be taken among the objective reals. But a wider view suggests that it is a protest against dividing the class of objective reals into two parts, one part (the space of the classical concept) being the field of fundamental relations which do not include instants of time in their fields, and the other part (the particles) only occurring in the fields of fundamental relations which do include instants of time. In this sense it is a protest against exempting any part of the universe from change. But it is not probable that this is the light in which

LEIBNIZ himself regarded the theory. This theory, though at present it is nominally the prevailing one, has never been worked out in the form of a precise mathematical concept. It is on this account criticized severely by BERTRAND RUSSELL (*cf. loc. cit.* and 'Philosophy of LEIBNIZ,' Cambridge, 1900, p. 120), who, however, has gone further than any of its upholders to give it mathematical precision. Of course, from the point of view of this paper, we are not concerned with upholding or combatting any theory of the material world. Our sole purpose is to exhibit concepts not inconsistent with some, if not all, of the limited number of propositions at present believed to be true concerning our sense-perceptions.

Definition.—Any concept of the material world which demands two classes of objective reals will be called a *Dualistic* concept; whereas a concept which demands only one such class will be called a *Monistic* concept.

The classical concept is dualistic; Leibnizian concepts will be, in general, monistic (*cf.* however Concept IVA.). OCCAM'S razor—Entia non multiplicanda præter necessitatem—formulates an instinctive preference for a monistic as against a dualistic concept. Concept III. below is an example of a Leibnizian monistic concept. The objective reals in it may be considered to represent either the particles or the points of the classical concept. But they change their spatial relations. Perhaps LEIBNIZ was restrained from assimilating his ideas more closely to Concept III. by a prejudice against anything, so analogous to a point of space, moving—a prejudice which arises from confusing the classical dualistic concept with the monistic concepts. It is of course essential that at least some members of the class of objective reals should have different relations to each other at different instants. Otherwise we are confronted with an unchanging world. Concept V. is another Leibnizian monistic concept.

The Time-Relation.—In every concept a dyadic serial relation, having for its field the instants of time and these only, is necessary. The properties of this *Time-Relation* form the pure science of chronology. The time-relation is, in all concepts, a serial relation ordinally similar to the serial relation which generates the series of negative and positive real numbers.* This fact need not be further specified during the successive consideration of the various concepts, nor need any of the propositions of pure chronology be enunciated.

Definition.—The class of instants of time is always denoted by T in every concept.

The Essential Relation.—In every concept at least all the propositions of geometry will be exhibited as properties of a single polyadic relation, here called the *essential relation.* The field of the essential relation will consist, *either* of the whole class of ultimate existents (*e.g.*, in Concepts III., IVB. and V.), *or* of part of the class of objective reals together with the instants of time (*e.g.*, in Concept IVA.), *or* of the whole class of objective reals (*e.g.*, in Concept II.), *or* of part of the class of objective

* For interesting reflections on this subject, influenced by the Kantian Philosophy and previous to the modern "Logicization of Mathematics," *cf.* HAMILTON, 'Lectures on Quaternions,' preface.

reals (*e.g.*, in Concept I.). The essential relation of any one concept will be a relation between a definite finite number of terms, for example, between three terms in Concepts I. and II., between four terms in Concept III., and between five terms in Concepts IV. and V.*

Definition.—In the exposition of every concept, the essential relation of that concept is denoted by R.

The Extraneous Relations.—In all the concepts here considered, other relations, here called the *extraneous relations*, will be required in addition to the time-relation and the essential relation. In Concepts I. and II. and IV. an indefinite (if not infinite) number of extraneous relations are required, determining the positions of particles. In Concepts III., IV. and V. one tetradic extraneous relation is required to determine the "kinetic axes" of reference for the measurement of velocity.

The time-relation, the essential relation and the extraneous relations form the fundamental relations of any concept in which they occur.

It will now be necessary to define geometry anew, since the previous definition has essential reference to the dualism of the classical concept. A proposition of geometry is any proposition (1) concerning the essential relation; (2) involving one, and only one, instant of time; (3) true for any instant of time.

In the classical concept everything is sacrificed to simplicity in reference to geometry, probably because it arose when geometry was the only developed science. The result is that, when the properties of matter are dealt with, an appalling number of extraneous relations are necessary.

Judged by OCCAM's principle, this class of extraneous relations forms a defect in Concepts I. and II. and IV. Also, in both forms of the classical concept (viz., in Concepts I. and II.) geometry is segregated from the other physical sciences to a greater degree than in the other concepts.

In the study of any concept there are four logical stages of progress. The first stage consists of the *definition* of those entities which are capable of definition in terms of the fundamental relations. These definitions are logically independent of any axioms concerning the fundamental relations, though their convenience and importance are chiefly dependent upon such axioms. The second stage consists of the

* The idea of deriving geometry (at least projective geometry without reference to order) from a single triadic relation was (I believe) first enunciated and investigated by Mr. A. B. KEMPE, F.R.S., in 1890, *cf.* "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," 'Proc. Lond. Math. Soc.,' vol. XXI. It has since been worked out in detail for Euclidean geometry by Dr. O. VEBLEN, *cf.* "A System of Axioms for Geometry," 'Trans. Amer. Math. Soc.,' vol. 5, 1904. Also *cf.* Professor J. ROYCE on "The Relation of the Principles of Logic to the Foundations of Geometry," 'Trans. Amer. Math. Soc.,' 1905. Professor ROYCE emphasises the importance of KEMPE'S work and considerably extends it. This memoir (which unfortunately only came into my hands after the completion of the present investigation) anticipates a general line of thought of the present paper in the emphasis laid on the derivation of geometry from a single polyadic relation; otherwise our papers are concerned with different problems.

deduction of those properties of the defined entities which do not depend upon the axioms. The third stage is the *selection* of the group of *axioms* which determines that concept of the material world. The fourth stage is the *deduction* of propositions which involve among their hypotheses some or all of the axioms of the third stage. Psychologically the order of study is apt to be inverted, by first choosing propositions of the second and fourth stages because of their parallelism with the propositions of sense-perception and then by considering the first and third stages. The essential part of our task in passing concepts in review is the exhibition of the first and third stages. The second and fourth stages will only be so far touched upon as seems desirable for the purposes of elucidation.

Thus in respect to each concept considered the investigation will proceed as follows:—A certain relation R (the essential relation of the concept in question), which holds between a certain definite number of entities, is considered. The class of entities, between sets of which this relation holds, is called the "field" of R. Definitions of entities allied to R and to entities of the field of R are then given. These definitions involve no hypotheses as to the properties of R, but are of no importance unless R has as a matter of fact certain properties. For example, it may happen that the classes, thus defined, are all the null class (*i.e.*, the class with no members) unless R has the requisite properties. Again deductions (in the second stage), made without any hypothesis as to the properties of R, may be entirely trivial unless R has certain properties. If R has not the requisite properties the deductions often sink into the assertion that a certain proposition which is false implies some other proposition. This is true* but trifling. The "axioms" respecting R are then given. These are the hypotheses as to the properties of R which are required in the concept under consideration. Finally such deductions are given as are necessary to elucidate the concept.

None of the reasoning of this memoir depends on any special logical doctrine which may appear to be assumed in the form in which it is set out. Furthermore certain contradictions recently discovered have thrown grave doubt upon the current doctrine of classes as entities. Any recasting of our logical ideas upon the subject of classes must of course simply issue in a change of our ideas as to the true logical analysis of propositions in which classes appear. The propositions themselves, except a few extreme instances which lead to contradictions, must be left intact. Accordingly the present memoir in no way depends upon any theory of classes.

The above considerations as to method must essentially hold for any investigation respecting axioms of geometry or of physics, viewed purely as deductive sciences, and apart from the question of experimental verification.

In Concepts I., II., and III. the members of the "field" of R are to be considered as points, except those members of the field which are instants of time. In these concepts the lines and planes are classes of points. In Concepts IV. and V. the

* *Cf.* RUSSELL, 'The Principles of Mathematics,' § 16.

members of the "field" of R, other than the instants of time, are to be considered as lines taken as simple entities. Points are classes of these simple lines. But the ordinary line of geometry which has parts and segments is a class of points, and so is the ordinary plane of geometry. In Concept III., which is Leibnizian and monistic, the points (perhaps "particles" is here a better word) move, and the straight lines and planes disintegrate from instant to instant. In Concepts IV. and V. the points similarly disintegrate.

(ii) Explanation of Symbolism.

This explanation is only concerned with the general logical symbolism. The special symbols which arise out of the ideas of the paper are defined in their proper places. PEANO'S* chief symbols are used. The changes and developments from PEANO, which will be found here, are due to RUSSELL and myself working in collaboration for another purpose. It would be impossible to disentangle our various contributions.†

None of the reasoning of the paper is based upon any peculiarity of the symbolism. It is used here only as an alternative form for enunciations, for the sake of its conciseness and (above all) its precision. In the verbal enunciations precision has been to some extent sacrificed to lucidity; and the exact statement of what is meant is always to be sought in the symbolic alternative form. The proofs have been translated into words out of the symbolic form in which they were mostly elaborated.

On $\supset$, $\equiv$, $\subset$, ϵ, $=$, $=$ Df

There are five copulas, namely, $\supset$, $\equiv$, $\subset$, ϵ, $=$. Here $x \supset y$ means *x implies y;* and $x \equiv y$ means *x implies y and y implies x;* and $x \subset y$ means *all x's are y's;* and $x \epsilon y$ means *x is a member of y;* and $x, y \epsilon u$ means *x and y are members of u.* Note that $x \subset y$ implies that x is a subclass of y; a class will be said to *contain* a subclass and to *possess* a member. Lastly, $x = y$ means *x is identical with y.* Note that, if Df, short for Definition, is placed at the end of the line, thus,

$$x = y \qquad \text{Df}$$

the symbols mean that *x is defined to stand for y.* In such a case y is some complex of symbols, and x will be an abbreviated symbol standing for y.

On $\phi!x$, $(x).\phi!x$, $(\exists x).\phi!x$, (x, y), $(\exists x, y)$, $\supset_x$

Propositional Functions.—$\phi!x$ means *x has the property* ϕ, where ϕ is given different forms corresponding to different properties; $\exists!x$ means *x has the property*

* *Cf.* 'Notations de Logique Mathématique,' Turin, 1894; and 'Formulaire Mathématique,' Turin, 1903.

† See, however, RUSSELL'S articles, "Sur la Logique des Relations," 'Revue de Mathématiques,' vol. VII., 1900–1901, Turin.

of being a class possessing at least one member; and $(x) . \phi ! x$ means $\phi ! x$ *is true for every value of* x*;* and $(\exists x) . \phi ! x$ means *there exists a value of* x *for which* $\phi ! x$ *is true.* Note that (x) and $(\exists x)$, written before any proposition involving x, give the above meanings, even if the proposition is not in the symbolic form $\phi ! x$. If the proposition involve both x and y, then (x, y) prefixed means that the proposition is true for all values of x and y; and similarly for (x, y, z), and so on. Also $(\exists x, y)$ prefixed means that there exist values of x and y, such that the proposition is true; and similarly for $(\exists x, y, z)$, and so on. Furthermore $\phi ! x \supset_x \psi ! x$ stands for $(x) . \{\phi ! x \supset \psi ! x\}$, and similarly for two and three variables.

On the Use of Dots, viz., $.$, $:$, $.\!:$, $::$

$p . q$ or $p : q$ or $p \therefore q$ or $p :: q$ all mean p *and* q *are both true propositions.* As an example, $x, y \in u$, which has been defined above, is really the proposition $x \in u . y \in u$; and $x, y, z \in u$ is the proposition $x \in u . y \in u . z \in u$.

Dots as Brackets.—The different symbolic forms for the joint assertion of propositions arise from the fact that dots are also used as bracket forms for propositions according to the following rules:—

(i) The larger aggregation of dots represents the exterior bracket. (ii) The dots at the end of a complete sequence of symbols are omitted. (iii) The dots immediately preceding or succeeding the implication sign, viz., $\supset$, are exterior brackets to any equal number of dots occurring in other capacities (*e.g.*, as above in the joint assertion of propositions). (iv) The dots which also serve to indicate the joint assertion of propositions are interior brackets to any equal number of dots occurring in other capacities. (v) The dots after (x) and $(\exists x)$ are increased in number according to the necessity for their use as brackets.

In reading a symbolic proposition it is best to begin by searching for that implication sign, viz., $\supset$, which is preceded or succeeded by the greatest number of dots. This splits up the proposition into hypothesis and consequent; and so on with these subsidiary propositions, if necessary.

On $\vee$, $\sim$, $\sim\in$, $\neq$, $\sim(x) . \phi ! x$, $\sim(\exists x) . \phi ! x$

Again $p \vee q$ means *one or other or each of* p *and* q *is a true proposition;* and $\sim p$ means p *is not true.* Thus $\sim \phi ! x$ means x *has not the property* ϕ; also $x \sim\in u$ stands for $\sim(x \in u)$; and $x \neq y$ stands for $\sim(x = y)$; and $\sim(x) . \phi ! x$ stands for $\sim\{(x) . \phi ! x\}$; and $\sim(\exists x) . \phi ! x$ stands for $\sim\{(\exists x) . \phi ! x\}$.

On $\hat{x}(\phi ! x)$, $(\imath x)(\phi ! x)$, $\iota‘$, $\imath‘$, $\cup$, $\cap$, $\cup‘$, $\cap‘$

Non-Propositional Functions.—$\hat{x}(\phi ! x)$ denotes *the class of terms which have the property* ϕ, and $(\imath x)(\phi ! x)$ denotes *the single entity* (*if there is such*) *which, when*

substituted for x, makes $\phi!x$ to be a true proposition. It is not necessary for the above symbolism that the proposition involving x should be in the symbolic form $\phi!x$. Again, $\iota`x$ denotes *the class possessing x as its sole member*, and $\text{\rotatebox{180}{$\iota$}}`x$ denotes *the sole member of the class x*, and $u \cup v$ denotes *the logical sum of u and v*, that is, the class possessing all members of u and all members of v and no other members. Thus, $\iota`a \cup \iota`b$ denotes the class whose sole members are a and b. Again, $u \cap v$ denotes *the logical product of u and v*, that is, the complete common subclass of u and v; and $\cup`u$ denotes *the class which is the logical sum of all members of u*, that is, the class which has as members all members of members of $u$; and $\cap`u$ denotes *the class which is the logical product of all members of u.* The exact symbolic definition of $\cap`u$ is

$$\cap`u = \hat{x}\,\{v \in u \,.\supset_v .\, x \in v\} \qquad \text{Df}$$

It follows from this definition that, if u possess no members, $\cap`u$ is the class of all entities.

On Λ, cls‘, $-$, Nc‘

Again, Λ denotes the *null class*, that is, the class with no members; cls‘u denotes *the class whose members are the subclasses contained in u, including u itself and the null class.* It follows that the propositions, $v \in \text{cls}`u$ and $v \subset u$, have practically identical meanings. Again, $u - v$ denotes *the class u with the exception of those members which it possesses in common with v.*

The cardinal numbers* are themselves classes. Thus, 1 is the class whose members are the unit classes, 2 is the class whose members are couples. Accordingly, $x \in 2$ means *x is a class with two members*; Nc‘u denotes *the cardinal number of the class u.*

On $\phi`$, $\phi``$, $\iota``$, $\text{\rotatebox{180}{$\iota$}}``$, $\cup``$, *and so on.*

The general form for a non-propositional function whose value depends on x is $\phi`x$, where ϕ receives different forms for different functions, as has been illustrated by the particular cases considered above. The apostrophe may be read as "of"; it is the general symbol for the connection of the preceding functional sign with the succeeding argument. According to this rule we should write sin‘x for sin x and log‘x for log x. Again, $\phi``u$ denotes *the class of values of $\phi`x$, when the various members of u are substituted for x*; it may be read "the class of $\phi$'s of $u$'s." Thus, if $\phi`x$ is "the head of $x$," and $u$ is "the class of horses," then $\phi``u$ is "the class of heads of horses." The exact symbolic definition of $\phi``u$ is as follows:

$$\phi``u = \hat{z}\,\{(\exists x)\,.\,x \in u\,.\,z = \phi`x\} \qquad \text{Df}$$

It follows from the definition, by substituting for ϕ, that $\iota``u$, $\text{\rotatebox{180}{$\iota$}}``u$, $\cup``u$, $\cap``u$, cls‘‘u, Nc‘‘u are now defined.

* *Cf.* RUSSELL, 'Principles of Mathematics,' chap. XI., and FREGE, 'Grundlagen der Arithmetik,' Breslau, 1884, pp. 79, 85.

On $(\mathrm{E}x)(\phi\text{‘}y)$

Again, $(\mathrm{E}x)(\phi\text{‘}y)$ means *there exists an entity which is denoted by the non-propositional function* $\phi\text{‘}z$, *when* z *has the particular value* y. For example, if u is a class, there is such an entity as its cardinal number, denoted by $\mathrm{N}c\text{‘}u$; but if u is not a class, there is no such entity as its cardinal number.*

On Symbols of the Type $\mathrm{R}^{;}(\quad)$.

Relations.—$\mathrm{R}^{;}(xyz)$ means x, y, z *form an instance in which the triadic relation* R *holds, the special "positions" of* x, y, z *in this instance of that relationship being indicated by their order of occurrence in the symbol* $\mathrm{R}^{;}(xyz)$. Again, $\mathrm{R}^{;}(\cdot yz)$ means *there exists an entity,* x *say, such that* $\mathrm{R}^{;}(xyz)$. The symbolic definitions of $\mathrm{R}^{;}(\cdot yz)$, and of analogous symbols, are

$$\mathrm{R}^{;}(\cdot yz) . = . (\exists x) . \mathrm{R}^{;}(xyz) \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(x \cdot z) . = . (\exists y) . \mathrm{R}^{;}(xyz) \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(xy \cdot) . = . (\exists z) . \mathrm{R}^{;}(xyz) \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(x \cdot\cdot) . = . (\exists y, z) . \mathrm{R}^{;}(xyz) \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(\cdot y \cdot) . = . (\exists x, z) . \mathrm{R}^{;}(xyz) \qquad \mathrm{Df}$$

and so on. Again, $\mathrm{R}^{;}(;yz)$ denotes *the class of terms, such as* x *(say), which satisfy* $\mathrm{R}^{;}(xyz)$, and $\mathrm{R}^{;}(\cdot;z)$ denotes *the class of terms, such as* y *(say), such that there exists a term,* x *say, such that* $\mathrm{R}^{;}(xyz)$ *holds.* The symbolic definitions of $\mathrm{R}^{;}(;yz)$ and of analogous entities, and of $\mathrm{R}^{;}(\cdot;z)$ and of analogous entities, are

$$\mathrm{R}^{;}(;yz) = \dot{x}\,\{\mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(x;z) = \dot{y}\,\{\mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(xy;) = \dot{z}\,\{\mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(;\cdot z) = \dot{x}\,\{(\exists y) . \mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(\cdot;z) = \dot{y}\,\{(\exists x) . \mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(;y\cdot) = \dot{x}\,\{(\exists z) . \mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$

and so on:

$$\mathrm{R}^{;}(;\cdot\cdot) = \dot{x}\,\{(\exists y, z) . \mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(\cdot;\cdot) = \dot{y}\,\{(\exists x, z) . \mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$
$$\mathrm{R}^{;}(\cdot\cdot;) = \dot{z}\,\{(\exists x, y) . \mathrm{R}^{;}(xyz)\} \qquad \mathrm{Df}$$

* The difficult question of the import of a proposition, which contains a non-propositional function (with some particular entity as argument) to which no entity corresponds, has recently been elucidated by RUSSELL, *cf.* 'Mind,' October, 1905. All propositions containing such a function are untrue, unless the function is merely a constituent of a subsidiary proposition whose truth is not implied by the proposition in question.

Thus, $R^{\cdot}(;\cdot\cdot)$ denotes *the class of terms, such as* x *(say), such that there exist terms,* y *and* z *say, such that* $R^{\cdot}(xyz)$ *holds.*

Again, $R^{\cdot}(;;z)$ denotes *the class which is the logical sum of* $R^{\cdot}(;\cdot z)$ *and* $R^{\cdot}(\cdot;z)$; and $R^{\cdot}(;;\cdot)$ denotes *the class which is the logical sum of* $R^{\cdot}(;\cdot\cdot)$ *and* $R^{\cdot}(\cdot;\cdot)$; and $R^{\cdot}(;;;)$ denotes *the class which is the logical sum of* $R^{\cdot}(;\cdot\cdot)$ *and* $R^{\cdot}(\cdot;\cdot)$ *and* $R^{\cdot}(\cdot\cdot;)$, that is, the "field" of the relation R. The symbolic definitions of the above, and of similar entities, are:

$$R^{\cdot}(;;z) = R^{\cdot}(;\cdot z) \cup R^{\cdot}(\cdot;z) \quad \text{Df}$$

$$R^{\cdot}(;y;) = R^{\cdot}(;y\cdot) \cup R^{\cdot}(\cdot y;) \quad \text{Df}$$

$$R^{\cdot}(x;;) = R^{\cdot}(x;\cdot) \cup R^{\cdot}(x\cdot;) \quad \text{Df}$$

$$R^{\cdot}(;;\cdot) = R^{\cdot}(;\cdot\cdot) \cup R^{\cdot}(\cdot;\cdot) \quad \text{Df}$$

and so on, and

$$R^{\cdot}(;;;) = R^{\cdot}(;\cdot\cdot) \cup R^{\cdot}(\cdot;\cdot) \cup R^{\cdot}(\cdot\cdot;) \quad \text{Df}$$

This notation, which has been explained for triadic relations, can obviously be extended to any polyadic relations. Thus, $R^{\cdot}(abcd)$ and $R^{\cdot}(abcdt)$ are defined in a similar manner, and so are the symbols for the allied propositions and classes.

On $1 \rightarrow 1$.

A dyadic relation, S say, is called *one-one*, when each referent has only one relatum, and each relatum has only one referent. The class of one-one relations is denoted by $1 \rightarrow 1$. The symbolic definition is

$$1 \rightarrow 1 \,.=.\, \dot{S}\,\{S \in \text{relation} : x \in S^{\cdot}(;\cdot) \,.\supset_x.\, S^{\cdot}(x;) \in 1 : y \in S^{\cdot}(\cdot;) \,.\supset_y.\, S^{\cdot}(;y) \in 1\} \quad \text{Df}$$

On $\vdash$

The Assertion Sign.—A proposition, which is stated in symbols as being true, *i.e.*, which is *asserted* as distinct from being *considered*, has the symbol $\vdash$ prefixed to it, with as many dots following as will serve to bracket off the proposition. This symbol $\vdash$ is called the *assertion sign.**

Part II.—The Punctual Concepts.

Those concepts of the material world in which the class of objective reals is composed of *points*, or *particles*, or of *both*, will be called the *punctual* concepts. The classical concept is a punctual concept, and will be considered first. The other punctual concepts can be explained briefly by reference to the classical concept.

Concept I.—(*The Classical Concept*).—This is dualistic, the class of objective reals

* This symbol is due to Frege, who first drew attention to the necessity of the idea which it symbolizes; *cf.* his 'Begriffschrift,' Halle, 1879, and Russell, 'Principles of Mathematics,' p. 35.

being subdivided into *points of space* and *particles of matter*. The essential relation has for its field the points of space only. Slight variants (not considered here) can be given to the concept by varying the properties of the essential relation, so as to make the geometry non-Euclidean, or, retaining Euclidean geometry, so as to give various forms to the essential relation and the resulting axioms. In the exposition of a system of geometrical axioms for Concept I., VEBLEN'S memoir (*cf. loc. cit.*), to which I am largely indebted, will be followed. The changes which are made from VEBLEN'S treatment are (i) in the addition of the symbolism which emphasizes the idea of the essential relation, and (ii) in the fact that the question of the independence of axioms is here ignored, through a desire not to overload this memoir with difficulties (both for the author and reader) belonging to another part of the subject. As the result of (ii), some of VEBLEN'S definitions and axioms have been simplified (and, in a sense, spoiled). The axioms thus obtained for Concept I. will shorten our investigations of other concepts by serving as a standard of comparison to determine whether the axioms of the other concepts are sufficient to yield three-dimensional Euclidean geometry.*

The essential relation (called R) is triadic. $\mathrm{R}\text{;}(abc)$ means *the points a, b, c are in the linear order* (*or the* R-*order*) *abc*. The relation R, when $\mathrm{R}\text{;}(abc)$ holds, is not symmetrical as between the three points a, b, and c; namely, it will be found that a and b (or b and c) cannot be interchanged.

Definitions of Concept I.

Definition.—The class $\mathrm{R}\text{;}(a;b)$ is the *segment between a and b*; and the class $\mathrm{R}\text{;}(ab;)$ is the *segmental prolongation of ab beyond b*; and the class $\mathrm{R}\text{;}(;ab)$ is the *segmental prolongation of ab beyond a*. It follows from the subsequent axioms that $\mathrm{R}\text{;}(ab;)$ is identical with $\mathrm{R}\text{;}(;ba)$.

Definition.—The *straight line ab* is the logical sum of $\mathrm{R}\text{;}(a;b)$ and $\mathrm{R}\text{;}(;ab)$ and $\mathrm{R}\text{;}(ab;)$ together with a and b themselves. Its symbol is $\mathrm{R}\text{;}\overline{ab}$. The definition in symbols is

$$\mathrm{R}\text{;}\overline{ab} = \mathrm{R}\text{;}(;ab) \cup \mathrm{R}\text{;}(a;b) \cup \mathrm{R}\text{;}(ab;) \cup \iota\text{‘}a \cup \iota\text{‘}b \qquad \text{Df}$$

* On the philosophical questions connected with the mathematical analysis of geometry *cf.* 'A Critical Exposition of the Philosophy of LEIBNIZ,' Cambridge, 1900, and 'The Principles of Mathematics,' Cambridge, 1903, both by BERTRAND RUSSELL; and also two articles by L. COUTURAT in the 'Revue de Métaphysique et de Morale' (Paris) for May and September, 1904, one entitled "La philosophie des Mathématiques de KANT" and the other "Les principes des Mathématiques—VI. La géometrie"; also POINCARÉ'S 'Science and Hypothesis,' Part II., English translation, London, 1905.

For expositions of exact systems of axioms *cf.* 'Vorlesungen über neuere Geometrie,' Leipzig, 1882, by PASCH; also 'I Principii di Geometria,' Turin, 1889, by PEANO; also "I Principii della Geometria di Posizione," 'Trans. Acad. of Turin,' 1898, by PIERI; also HILBERT'S 'Foundations of Geometry,' Engl. Transl., Chicago, 1902; also Professor E. H. MOORE, "On the Projective Axioms of Geometry," 'Transact. of the Amer. Math. Soc.,' 1903; also Dr. O. VEBLEN (*loc. cit.*); also Professor J. ROYCE (*loc. cit.*).

Definition.—The class whose members are the various straight lines is denoted by lin_R. The definition in symbols is

$$\text{lin}_R = v\{(\exists a, b) \,.\, a, b \in R^{\cdot}(;;;) \,.\, a \neq b \,.\, v = R^{\cdot}\overline{ab}\} \qquad \text{Df}$$

Definition.—Any class of points [*i.e.*, members of $R^{\cdot}(;;;)$] is called a *figure*.

Definition.—The *class of lines defined by a figure u* is the class of lines defined by any two distinct points of u. Its symbol is $\text{ln}_R{}^{\backprime}u$. The definition in symbols is

$$\text{ln}_R{}^{\backprime}u = \hat{v}\{(\exists x, y) \,.\, x, y \in u \cap R^{\cdot}(;;;) \,.\, x \neq y \,.\, v = R^{\cdot}\overline{xy}\} \qquad \text{Df}$$

Definition.—The *linear figure defined by a figure u* is the logical sum of all the lines defined by u (*i.e.*, is all the points on such lines). Thus its symbol is $\cup{}^{\backprime}\text{ln}_R{}^{\backprime}u$.

Definition.—Three points form a triangle, if there is no line which possesses them all. The symbol expressing that a, b, c are points forming a triangle is $\Delta_R{}^{\cdot}(abc)$. The definition in symbols is

$$\Delta_R{}^{\cdot}(abc) \,.\, = \,.\, a, b, c \in R^{\cdot}(;;;) \,.\, \sim (\exists v) \,.\, v \in \text{lin}_R \,.\, a, b, c \in v \qquad \text{Df}$$

Definition.—The *space defined by the triangle abc* is the linear figure defined by the linear figure defined by the three points a, b, c. Its symbol is $\Pi_R(abc)$. The definition in symbols is

$$\Pi_R(abc) = \cup{}^{\backprime}\text{ln}_R{}^{\backprime}\cup{}^{\backprime}\text{ln}_R{}^{\backprime}(\iota{}^{\backprime}a \cup \iota{}^{\backprime}b \cup \iota{}^{\backprime}c) \qquad \text{Df}$$

Definition.—The *class of planes* is the class of spaces defined by any three points a, b, c when they form a triangle. Its symbol is ple_R. The definition in symbols is

$$\text{ple}_R = \hat{v}\{(\exists a, b, c) \,.\, \Delta_R{}^{\cdot}(abc) \,.\, v = \Pi_R(abc)\} \qquad \text{Df}$$

Definition.—The space defined by the four points a, b, c, d is the linear figure defined by the figure which is the logical sum of $\Pi_R(bcd)$ and $\Pi_R(acd)$ and $\Pi_R(abd)$ and $\Pi_R(abc)$. Its symbol is $\Pi_R(abcd)$. The symbolic definition is

$$\Pi_R(abcd) = \cup{}^{\backprime}\text{ln}_R{}^{\backprime}\{\Pi_R(bcd) \cup \Pi_R(acd) \cup \Pi_R(abd) \cup \Pi_R(abc)\} \qquad \text{Df}$$

The above definitions are sufficient to exhibit the dependence of the various geometrical entities on the essential relation, and also to enable us, as far as geometry is concerned, to pass on to the third stage. Owing to the simplicity of the definitions, the second stage for this concept is of very small importance.

It will be noticed that none of the definitions contain any reference to length, distance, area, or volume. This is because none of these ideas appear in the axioms, and only such definitions are given here as are necessary for the enunciation of the axioms. According to the well-known* method of projective metrics, the ideas are introduced by definition and require no special axiom.

* *Cf.* VEBLEN, *loc. cit.*; also 'Vorlesungen über Geometrie,' by CLEBSCH, third part; also 'The Principles of Mathematics,' by RUSSELL, chap. XLVIII.

Axioms of Concept I.

The axioms, it must be remembered, are merely an enumeration of various propositions concerning the properties of the fundamental relations, which will occur as hypotheses in the propositions of the fourth stage. In this instance we are merely considering the axioms of geometry, and these concern the essential relation (R) only. The axioms will be named systematically thus, I H*p* R, II H*p* R, III H*p* R, and so on. Their enumeration will take the form of defining these *names* as abbreviations standing for the various statements, which will be used subsequently as hypotheses.

I H*p* R is the statement that there is at least one set of entities, a, b, c, such that R$\text{'}(abc)$ is true. The definition in symbols is

$$\text{I Hp R} \,.=.\, \exists\,!\,\text{R}'(;;;) \qquad \text{Df}$$

II H*p* R is the statement that R$'(abc)$ implies R$'(cba)$. The definition in symbols is

$$\text{II Hp R} \,.=:\, (a, b, c) : \text{R}'(abc) \,.\supset.\, \text{R}'(cba) \qquad \text{Df}$$

III H*p* R is the statement that R$'(abc)$ is inconsistent with R$'(bca)$. The definition in symbols is

$$\text{III Hp R} \,.=:\, (a, b, c) : \text{R}'(abc) \,.\supset.\, \sim \text{R}'(bca) \qquad \text{Df}$$

IV H*p* R is the statement that R$'(abc)$ implies that a is distinct from c. The definition in symbols is

$$\text{IV Hp R} \,.=:\, (a, b, c) : \text{R}'(abc) \,.\supset.\, a \neq c \qquad \text{Df}$$

V H*p* R is the statement that, if a and b are distinct points, the segmental prolongation of ab beyond b possesses at least one member. The definition in symbols is

$$\text{V Hp R} \,.=:\, (a, b) : a, b \in \text{R}'(;;;) \,.\, a \neq b \,.\supset.\, \exists\,!\{\text{R}'(ab;)\} \qquad \text{Df}$$

VI H*p* R is the statement that, if c and d are distinct points, possessed by the line defined by the points a and b, then a is possessed by the line defined by c and d. The definition in symbols is

$$\text{VI Hp R} \,.=:\, (a, b, c, d) : c, d \in \text{R}'\overline{ab} \,.\, c \neq d \,.\supset.\, a \in \text{R}'\overline{cd} \qquad \text{Df}$$

VII H*p* R is the statement that there exist at least three points forming a triangle. The definition in symbols is

$$\text{VII Hp R} \,.=.\, (\exists\, a, b, c) \,.\, \Delta_{\text{R}}'(abc) \qquad \text{Df}$$

VIII H*p* R is the statement that, if a, b, c be three points forming a triangle, and R$'(bcd)$ and R$'(cea)$ hold, then there exists a point possessed both by the segment ab, and by the line defined by d and e. The definition in symbols is

$$\text{VIII Hp R} \,.=:\, (a, b, c, d, e) : \Delta_{\text{R}}'(abc) \,.\, \text{R}'(bcd) \,.\, \text{R}'(cea) \,.\supset.\, \exists\,!\{\text{R}'\overline{de} \cap \text{R}'(a;b)\} \qquad \text{Df}$$

IX Hp R is the statement that there exists a point and a plane, such that the plane does not possess the point. The definition in symbols is

$$\text{IX Hp R} \,.=:\, (\exists\, p, d) \,.\, p \,\epsilon\, \text{ple}_{\text{R}} \,.\, d \,\epsilon\, \text{R}^{\prime}(;;;) - p \qquad \text{Df}$$

X Hp R is the statement that there exist four points a, b, c, d, such that the three-dimensional space $\Pi_{\text{R}}(abcd)$ contains the whole class of points. The definition in symbols is,

$$\text{X Hp R} \,.=.\, (\exists\, a, b, c, d) \,.\, \text{R}^{\prime}(;;;) \subset \Pi_{\text{R}}(abcd) \qquad \text{Df}$$

XI Hp R is some statement which secures the continuity (in CANTOR's sense) of the points on a line. The axiom need not be given here, since there will be no reasoning in this memoir connected with it.

XII Hp R is the statement that, if α be any plane and a a line contained in it, then there exists a point c in α, such that there is not more than one line, possessing c and contained in the plane α and not intersecting a. The definition in symbols is

$$\text{XII Hp R} \,.=.\therefore\, \alpha \,\epsilon\, \text{ple}_{\text{R}} \,.\, a \,\epsilon\, \text{lin}_{\text{R}} \cap \text{cls}\text{‘}\alpha \,.\supset_{a,\alpha}:\, (\exists\, c) : c \,\epsilon\, \alpha :$$
$$l, l^{\prime} \,\epsilon\, \text{lin}_{\text{R}} \cap \text{cls}\text{‘}\alpha \,.\, c \,\epsilon\, l \cap l^{\prime} \,.\, l \cap a = \Lambda \,.\, l^{\prime} \cap a = \Lambda \,.\supset_{l,l^{\prime}} .\, l = l^{\prime} \qquad \text{Df}$$

Of these axioms, IX Hp R secures that space is of three dimensions at least, and X Hp R secures that it is of three dimensions at most, and XII Hp R is the "Euclidean" axiom. From these twelve axioms the whole of geometry* can be deduced. The well-known parabolic (*i.e.*, Euclidean) *definition* of distance (not given here) assumes an important significance, and all the usual metrical properties follow.

The Extraneous Relations.—Nothing could be more beautiful than the above issue of the classical concept, if only we limit ourselves to the consideration of an unchanging world of space. Unfortunately, it is a changing world to which the complete concept must apply, and the intrusion at this stage into the classical concept of the necessity of providing for change can only spoil a harmonious and complete whole. Owing to the fact that the instants of time are not members of the field of the essential relation, the time relation and the essential relation have (so to speak) no point of contact. To remedy this, another subdivision of the class of objective reals is conceived, namely, the class of *particles* (where the particles are the ultimate entities composing the fundamental "stuff" which moves in space). These particles must form part of the fields of a class of extraneous relations. Each such extraneous relation is conceived as a triadic relation, which in any particular instance holds between a particle, a point of space, and an instant of time. Also the field of each such extraneous relation only possesses one particle, and no particle belongs to the field of two such relations. Thus each extraneous relation is peculiar to one particle. Also, as has been pointed out by RUSSELL,† to whom the above analysis of these extraneous relations of the classical concept is in substance due, the impenetrability

* *Cf.* VEBLEN, *loc. cit.*

† *Cf.* 'Principles of Mathematics,' vol. I., § 440.

of matter is secured by the axiom that two different extraneous relations cannot both relate the same instant of time to the same point. The general laws of dynamics, and all the special physical laws, are axioms concerning the properties of this class of extraneous relations.

Thus the classical concept is not only dualistic, but has to admit a class of as many extraneous relations as there are members of the class of particles.

Instead of the specific relations of occupation for the various particles, one general triadic relation of occupation can be considered. Thus, $\mathrm{O}{}^{\prime}(p\mathrm{A}t)$ may be considered as the statement the particle p occupies the point A at the instant t. Then for any given A and t there is either one only or no particle p for which $\mathrm{O}{}^{\prime}(p\mathrm{A}t)$ is true. Then the laws of physics are the properties of this single extraneous relation O. But the use of this single relation apparently introduces no real simplification, differing in this respect from the use of the essential relation which so simplifies the statement of the axioms of geometry. The general relation O remains a mere alternative statement of the facts respecting the various specific relations of occupation.

Concept II.—This concept is a monistic variant of the classical concept suggested by RUSSELL.* In the classical concept the particles only occur as terms in the triadic extraneous relations. If we abolish the particles (in the "classical" sense), and transform the extraneous relations into dyadic relations between points of space and instants of time, everything will proceed exactly as in the classical concept. The reason for the original introduction of "matter" was, without doubt, to give the senses something to perceive. If a relation can be perceived, this Concept II. has every advantage over the classical concept. Otherwise the material world, as thus conceived, would appear to labour under the defect that it can never be perceived. But this is a philosophic question with which we have no concern.

Concept III.—This is a Leibnizian concept, and also a monistic variant of the classical concept, obtained by abandoning the prejudice against points moving.

This concept can be otherwise considered, as obtained from the modern (and Cartesian) point of view of the ether, as filling all space. The particles of ether (or moving points) compose the whole class of objective reals. The essential relation (R) is a tetradic relation, and, in each specific instance of the relation holding, three of the terms are objective reals and the remaining term is an instant of time. $\mathrm{R}{}^{\prime}(abct)$ may be read as stating *the objective reals a, b, c are in the* R-*order abc at the instant t*. Instead of $\mathrm{R}{}^{\prime}(abct)$, it will be convenient to write $\mathrm{R}_t{}^{\prime}(abc)$. Then the geometrical definitions are exactly those of Concept I., replacing R everywhere by R_t. Also the geometrical axioms are those of Concept I.; except (i) that R is replaced by R_t, (ii) that I Hp R, and VII Hp R, and IX Hp R, are further modified by the introduction of the hypothesis $t \in \mathrm{T}$—thus I Hp R of Concept I. becomes

$$\mathrm{I\,H}p\,\mathrm{R}\,. = : t \in \mathrm{T}\,. \supset_t .\ \exists\,!\,\mathrm{R}_t{}^{\prime}(;;:) \qquad \mathrm{Df}$$

* *Cf.* 'Principles of Mathematics,' vol. I., § 441.

and similarly for the other two axioms, and (iii) that one additional axiom (the axiom of persistence) must be introduced, namely,

$$\text{XIII Hp R} . = : t \in \text{T} . \supset_t . \text{R}^{\text{;}}(;;;\cdot) \subset \text{R}^{\text{;}}(;;;t) \qquad \text{Df}$$

This axiom of persistence is unnecessary for the geometrical reasoning, but is an integral part of the "physical" side of the concept. Also the hypothesis $t \in \text{T}$, which is introduced in the three axioms (I, VII, IX) Hp R, is unnecessary in the other axioms, since it is implied by the hypotheses already existing. The same explanation holds of the absence of the hypothesis, $t \in \text{T}$, from many axioms and propositions of subsequent concepts.

Thus at each instant the objective reals may be considered as the points of the classical concept, and the whole of Euclidean geometry holds concerning them. But at another instant the points will not have preserved the same geometrical relations as held between them at the previous instant. Thus, in the comparison of the states of the objective reals at different instants, the objective reals assume the character of particles.

The Extraneous Relation.—A single extraneous relation is necessary to obviate the difficulty of comparing straight lines and planes at one instant with similar entities at another instant. In what sense can a point at one instant be said to have the *same position* as a point at another instant? This definition can be effected by introducing into the concept a single tetradic extraneous relation S, so that, when $\text{S}^{\text{;}}(uvwt)$ holds, t is an instant of time, and u, v, w are intersecting straight lines mutually at right angles. Also corresponding to any instant t in the fourth term, there is one and only one line for each of the other terms respectively. This last condition, expressed in symbols, is

$$t \in \text{T} . \supset_t . \text{S}^{\text{;}}(;\cdot\cdot t) \in 1 . \text{S}^{\text{;}}(\cdot;\cdot t) \in 1 . \text{S}^{\text{;}}(\cdot\cdot;t) \in 1$$

The straight lines indicated at each instant by this relation are to be taken as the "kinetic axes."* Velocity and acceleration can now be defined, and a general continuity of motion (in some sense) must be included among the axioms.

This concept has the advantage over Concepts I. and II. that it has reduced the class of extraneous relations to one member only, in the place of the innumerable and perhaps infinite number of extraneous relations in the other two concepts. The concept pledges itself to explain the physical world by the aid of motion only. It was indeed a dictum with some eminent physicists of the nineteenth century that "motion is of the essence of matter." But this concept takes them rather sharply at their word. There is absolutely nothing to distinguish one part of the objective reals from another part except differences of motion. The "corpuscle" will be a volume in which some peculiarity of the motion of the objective reals exists and persists. Two

* *Cf.* W. H. MACAULAY, 'Bulletin of the Amer. Math. Soc.,' 1897.

different developments, viz., Concept IIIA. and Concept IIIB., are now possible, according as the persistence is taken to be of one or other of two possible types.

Concept IIIA.—Here the persistence is that of the *same objective reals* in the same special type of motion. KELVIN'S vortex ring theory of matter can be adapted to such a concept.

Concept IIIB.—Here the persistence is that of the *type of motion* in some volume, but not necessarily of the identity of the objective reals in the volume. The *continuity of motion* of a corpuscle as a whole becomes then the *definition of the identity* of a corpuscle at one instant with a corpuscle at another instant.

PART III.—(i) GENERAL EXPLANATIONS OF LINEAR CONCEPTS.

These concepts depart widely from the classical concept. The objective reals (at least those which, with the instants of time, form the field of the essential relation) have properties which we associate with straight lines, considered throughout their whole extent as single indivisible entities. These objective reals, which in Concept V. are all the objective reals, will be called *linear objective reals.* Perhaps, however, a closer specification of the linear objective reals of these concepts is to say that they are the lines of force of the modern physicist, here taken to be ultimate unanalysable entities which compose the material universe, and that geometry is the study of a certain limited set of their properties. But this mode of realizing the nature of the linear objective reals has also its pitfalls, for a line of force suggests *ends*, while these linear objective reals have no properties analogous to the properties of the ends of lines of force. The whole of a straight line, viewed as a point-locus, will be found to be associated with a linear objective real. The "linear" concepts here considered are all Leibnizian.

Concept IVA. is dualistic, and requires among the objective reals a class of "particles" in addition to the linear objective reals. Concept IVB. is the monistic variant of Concept IVA., obtained exactly as Concept II. is derived from Concept I. Both of the Concepts IVA. and IVB. labour under the same defect as Concepts I. and II. in requiring an indefinitely large class of extraneous relations. Concept V. is monistic, and is by far the most interesting of the set of linear concepts. It requires only one extraneous relation to perform a similar office to that of the extraneous relation in Concept III.

Points are now defined complex entities, being certain classes of linear objective reals. Geometers are already used to the idea of the point as complex. In projective geometry, as derived from descriptive geometry, the projective point is nothing but a class of straight lines.* This idea will now be extended to all

* *Cf.* PASCH, *loc. cit.*, and SCHUR, "Ueber die Einführung der sogenannten idealen Elemente in die projective Geometrie," 'Math. Annal.,' vol. XXXIX., and BONOLA, "Sulla Introduzione degli Enti improprii in Geometria projettiva," 'Giorn. di Mat.,' vol. XXXVIII.

points; and the descriptive point, from which in the current theory the projective point is ultimately derived, is here abolished. The "Theory of Interpoints" [*cf.* Part III. (ii)] and the "Theory of Dimensions" [*cf.* Part IV. (i)] represent two distinct methods of overcoming the following initial and obvious difficulty of these "linear" concepts:—A point is to be defined as the class of objective reals "concurrent" at a point. But this definition is circular. How can this circularity be removed? The Theory of Interpoints and the Theory of Dimensions give two separate answers to this question. The points in the linear concepts, being only classes of objective reals, are capable of disintegration. In fact, when motion is considered, it will be found that the points of one instant are, in general, different from the points of another instant, not in the sense of Concept III. that they are the same entities with different relations, but in the sense that they are different entities. More difficulty will probably be felt in conceiving anything analogous to a line as a simple unity. Here it is to be observed that a linear objective real does not replace a line of points of ordinary geometry. On the contrary, the class of those points (here called a *punctual line*), which have a given linear objective real as a common member, is this ordinary geometrical line. A punctual line has parts and segments in the ordinary way. The idea of a single unity underlying a straight line is not wholly alien to ordinary language. The idea of a direction, as it could also be used in non-Euclidian geometries where each line will have its own peculiar direction, may be conceived as being that of a line taken as a unit. But it is unnecessary to elaborate these considerations, as they have no relation to the logic of the subject.

In the dualistic Concept IVA. the particles form another class of objective reals in addition to the linear objective reals. Each particle is associated at each instant with some one point, that is, with some class of linear objective reals. Thus the two points, respectively associated at any instant with two particles, have in common one linear objective real. Thus, when mutually determined motions are considered, these linear objective reals assume the aspect of lines of force. In the monistic Concept V. the analogy of objective reals to lines of force arises in a similar way. In this case particles, in the sense used above, do not exist. Corpuscles, to use another term, are defined entities, analogous to the corpuscles of Concept III.; any general consideration of them is best deferred till the definitions can be understood.

In Concepts IV. and V. the conception of an ether is (in a sense) rendered unnecessary, or (in another sense) is largely modified. The collection of linear objective reals (*i.e.*, in Concepts IVB. and V., of *all* objective reals) now forms the entity (the ether) which "lies between" the corpuscles of gross matter. These corpuscles must be conceived as volumes with some peculiarity either of motion or of structure. Of course it might be found useful, for the explanation of physical phenomena, to assume that corpuscles of some sort are generally distributed between bodies of gross matter, thus forming an ether in a secondary sense. The ancient controversy concerning action at a distance becomes irrelevant in these concepts. In

one sense there is something, not mere space, between two distant corpuscles, namely, the objective reals possessed in common; in another sense there is a direct action between two distant corpuscles not depending on intervening corpuscles. In fact, the premises common to both bands of disputants are swept away.

The Essential Relation.—In both of the Concepts IV. and V. the essential relation (R) is a pentadic relation, and has for its field both the class of instants of time and that of linear objective reals, that is, in Concept V. the field is the complete class of ultimate existents. The proposition R‘(*abcdt*) can be read as the statement that *the objective real a intersects the objective reals b, c, d in the order bcd at the instant t.* This conception of "the intersection in order of three linear objective reals by a fourth at an instant of time" must be taken as a fundamental relation between the five entities. But the properties of the relation are not to be limited by the suggestion of the technical name "intersection." The axioms will be so assumed that R‘(*abcdt*) implies that *a*, *b*, *c*, and *d* are distinct. Also, when points are defined, it will be found that the axioms secure that *a* intersects *b*, *c*, and *d* in distinct points. Furthermore, in general, *b*, *c*, and *d* are *not* co-punctual; so that the case when *a* is a transversal of the pencil *b*, *c*, *d* of *co-punctual* lines is only a particular case of the satisfaction of R‘(*abcdt*).

Definitions.—The notation of the general symbolism provides us with the symbol R‘(;;;;·) for the class of linear objective reals, and with R‘(····;) for the class of instants. But these symbols are long. Accordingly O will be defined to stand for the class of linear objective reals, and T for the class of instants. Thus, in symbols,

$$\mathrm{O} = \mathrm{R}`(;;;;\cdot) \qquad \mathrm{Df}$$

$$\mathrm{T} = \mathrm{R}`(\cdot\cdot\cdot\cdot;) \qquad \mathrm{Df}$$

When "particles" (in Concept IV.) are not being directly considered, the term "objective real" will be used instead of "linear objective real," or "member of O."

(ii) The Theory of Interpoints.*

*1. The theory of *intersection-points* (shortened into *interpoints*) is required in both of the Concepts IV. and V. Accordingly, it is convenient to investigate it before the special consideration of either concept. In Concept IV. the interpoints are the points, and there are no other points. In Concept V. the interpoints are, in general, only portions of points, and a point may contain no interpoint or many interpoints. Thus

* From this point a continuous argument commences, and the sections and included propositions are numbered by a combined integral and decimal system, the whole number for the section and the decimal part for the proposition, also the symbol (*) is placed before an integral number marking a section. All the easier proofs of propositions are omitted, those proofs remaining being retained either as specimens, or as containing some point of difficulty. The omitted proofs are often replaced by references to the preceding propositions used in them, as a guide to their reconstruction. Note that "*cf.* *2·31·41·5" is used as a shortened form of "*cf.* *2·31 and *2·41 and *2·5."

the axioms of Concept IV. (*cf.* *2) and those of Concept V. (*cf.* *22) are two alternative sets of hypotheses as to the properties of R in connection with which the theory of interpoints, as given in the present *1, assumes importance. Some axioms, involving interpoints in their statements, are identical in Concept IV. and Concept V. These axioms are stated now in *1, and their simple consequences are deduced. The theory of interpoints depends on that of "similarity of position" in a relation. This general idea will only be explained in the special form in which it is here required in respect to the essential relation R.

*1·11. *Definition.*—An entity, y, will be said to have a *position* in the pentadic relation R, *similar* to that of the entity x, with a as first term and t as last (fifth) term, if, whenever the relation holds between five terms, a being the first term and t the last term, and either x or y or both occurring among the other terms, the relation also holds when x is substituted for y (whenever y occurs), and also holds when y is substituted for x (whenever x occurs). The symbol $\mathrm{R}`\left(\frac{a???t}{x}\right)$ denotes the class of entities with positions similar to that of x in the relation R, a being first term and t last term. The definition in symbols is

$$\mathrm{R}`\left(\frac{a???t}{x}\right) . = : \hat{y}\,\{(\xi, \eta) : \mathrm{R}`(ax\xi\eta t) . \equiv . \mathrm{R}`(ay\xi\eta t) :$$
$$\mathrm{R}`(a\xi x\eta t) . \equiv . \mathrm{R}`(a\xi y\eta t) : \mathrm{R}`(a\xi\eta xt) . \equiv . \mathrm{R}`(a\xi\eta yt)\} \quad \mathrm{Df}$$

*1·12. *Proposition.*—If y is a member of $\mathrm{R}`\left(\frac{a???t}{x}\right)$, then $\mathrm{R}`\left(\frac{a???t}{y}\right)$ is identical with $\mathrm{R}`\left(\frac{a???t}{x}\right)$. In symbols,

$$\vdash : y \,\epsilon\, \mathrm{R}`\left(\frac{a???t}{x}\right) . \supset . \mathrm{R}`\left(\frac{a???t}{y}\right) = \mathrm{R}`\left(\frac{a???t}{x}\right)$$

*1·13. *Proposition.*—x is a member of $\mathrm{R}`\left(\frac{a???t}{x}\right)$

*1·21. *Definition.*—A class P of objective reals is called an *intersection-point on a* (shortened into *interpoint on a*), when there exists an objective real x, which is a member of $\mathrm{R}`(a;;;t)$, and P is the class whose members are $a$ together with all the members of the class $\mathrm{R}`\left(\frac{a???t}{x}\right)$. The symbol $\mathrm{R}`(a???t)$ stands for *the class of interpoints on a at the instant t.* The definition in symbols is

$$\mathrm{R}`(a???t) = \dot{\mathrm{P}}\left\{(\exists x) . x \,\epsilon\, \mathrm{R}`(a;;;t) . \mathrm{P} = \iota`a \cup \mathrm{R}`\left(\frac{a???t}{x}\right)\right\} \quad \mathrm{Df}$$

*1·22. *Definition.*—P is called an *interpoint* of the relation R at the instant t, if there exists an objective real a, such that P is a member of $\mathrm{R}`(a???t)$. The symbol $\mathrm{intpnt}_{\mathrm{R}t}$ stands for *the class of interpoints of* R *at the instant t.* In symbols,

$$\mathrm{intpnt}_{\mathrm{R}t} = \dot{\mathrm{P}}\,\{(\exists a) . \mathrm{P} \,\epsilon\, \mathrm{R}`(a???t)\} \quad \mathrm{Df}$$

*1·23. *Proposition.*—If P and Q are distinct members of $R^{\backprime}(a???t)$, then a is the sole member common to P and Q. In symbols,

$$\vdash : P, Q \in R^{\backprime}(a???t) . P \neq Q . \supset . \iota^{\backprime}a = P \cap Q$$

Proof.—*Cf.* *1·11·12·21.

*1·31. *Definition.*—The interpoints B, C, D will be said to be in the *interpoint-order* BCD at the instant t with respect to the relation R, when there exist objective reals a, x, y, z, such that (1) B, C, D are members of $R^{\backprime}(a???t)$; (2) x is a member of B, y of C, z of D; (3) $R^{\backprime}(axyzt)$ holds. The symbol $R_{in}^{\backprime}(BCDt)$ stands for the statement that *the interpoints* B, C, D *are in the interpoint-order* BCD *at the instant t with respect to the relation* R. In symbols,

$$R_{in}^{\backprime}(BCDt) . = . (\exists a, x, y, z) . B, C, D \in R^{\backprime}(a???t) . x \in B . y \in C . z \in D . R^{\backprime}(axyzt) \qquad \text{Df}$$

*1·32. $\vdash . \text{intpnt}_{Rt} = R_{in}^{\backprime}(;;;t)$

Proof.—The class $R_{in}^{\backprime}(;;;t)$ is part (or all) of the class intpnt_{Rt} (*cf.* *1·31). Again (*cf.* *1·22), if B is a member of intpnt_{Rt}, objective reals a and x exist, such that x is a member of $R^{\backprime}(a;;;t)$, and B is the interpoint possessing a and x. Hence there are objective reals y and z, such that *either* $R^{\backprime}(axyzt)$ *or* $R^{\backprime}(ayxzt)$ *or* $R^{\backprime}(ayzxt)$. Hence (*cf.* *1·31), there are interpoints C and D, such that *either* $R_{in}^{\backprime}(BCDt)$ *or* $R^{\backprime}(CBDt)$ *or* $R^{\backprime}(CDBt)$. Hence B is a member of $R_{in}^{\backprime}(;;;t)$.

The theory of interpoints has its chief interest when the following axiom is satisfied. It is named intpnt Hp R.

*1·41. Intpnt Hp R is the statement that *if* A *be an interpoint at the instant t, and a be any member of* A, *then* A *is a member of* $R^{\backprime}(a???t)$. In symbols,

$$\text{Intpnt Hp R} . = : A \in \text{intpnt}_{Rt} . a \in A . \supset_{a, A, t} . A \in R^{\backprime}(a???t) \qquad \text{Df}$$

*1·42. *Proposition.*—Assuming intpnt Hp R, then if A and B are distinct members of intpnt_{Rt}, A and B have *either* no members in common *or* one only. In symbols,

$$\vdash .\because \text{intpnt Hp R} . \supset : A, B \in \text{intpnt}_{Rt} . A \neq B . \supset . A \cap B \in 0 \cup 1.$$

Proof.—*Cf.* *1·23·41.

The interest of the relation of interpoint-order (R_{in}) arises when the relation R satisfies four axioms specifying the idea that $R^{\backprime}(abcdt)$ expresses that a *intersects b, c, d in the order bcd.* These axioms (together with intpnt Hp R) will be employed both in Concept IV. and in Concept V. They will be named α Hp R, β Hp R, γ Hp R, δ Hp R.

*1·51. α Hp R is the statement that a *is not a member of* $R^{\backprime}(a;;;t)$. In symbols,

$$\alpha \text{ Hp R} . = . (a, t) . a \sim\in R^{\backprime}(a;;;t) \qquad \text{Df}$$

*1·52. β Hp R is the statement that $R^{\backprime}(abcdt)$ *implies* $R^{\backprime}(adcbt)$. In symbols,

$$\beta \text{ Hp R} . = : (a, b, c, d, t) : R^{\backprime}(abcdt) . \supset . R^{\backprime}(adcbt) \qquad \text{Df}$$

*1·53. γ Hp R is the statement that R'$(abcdt)$ *and* R'$(acdbt)$ *are inconsistent.* In symbols,

$$\gamma \,\mathrm{H}p\,\mathrm{R} \,.=: (a, b, c, d, t) : \mathrm{R}'(abcdt) \,.\supset.\, \sim \mathrm{R}'(acdbt) \qquad \mathrm{Df}$$

*1·54. δ Hp R is the statement that R'$(abcdt)$ *implies that* b *and* d *are distinct.* In symbols,

$$\delta \,\mathrm{H}p\,\mathrm{R} \,.=: (a, b, c, d, t) : \mathrm{R}'(abcdt) \,.\supset.\, b \neq d \qquad \mathrm{Df}$$

*1·61. *Proposition.*—Assuming $(\alpha, \beta, \gamma, \delta)$ Hp R, then R'$(abcdt)$ implies that a, b, c, d are all distinct. In symbols,

$$\vdash .\therefore (\alpha, \beta, \gamma, \delta)\,\mathrm{H}p\,\mathrm{R} \,.\supset: \mathrm{R}'(abcdt) \,.\supset.\, a \neq b \,.\, a \neq c \,.\, a \neq d \,.\, b \neq c \,.\, b \neq d \,.\, c \neq d$$

*1·62. *Proposition.*—Assuming $(\alpha, \beta, \gamma, \delta)$ Hp R, then $\mathrm{R_{in}}$'(BCDt) implies that B, C, D are all distinct. In symbols,

$$\vdash .\therefore (\alpha, \beta, \gamma, \delta)\,\mathrm{H}p\,\mathrm{R} \,.\supset: \mathrm{R_{in}}'(\mathrm{BCD}t) \,.\supset.\, \mathrm{B} \neq \mathrm{C} \,.\, \mathrm{B} \neq \mathrm{D} \,.\, \mathrm{C} \neq \mathrm{D}$$

Proof.—By definition (*cf.* *1·31) $\mathrm{R_{in}}$'(BCDt) implies that a, x, y, z exist such that B, C, D are members of R'$(a???t)$, x is a member of B, y of C, z of D, and R'$(axyzt)$. Hence (*cf.* *1·61) a, x, y, z are distinct. Now if any two of B, C, D are identical, *e.g.*, B and C, then x and y are both members of B. Hence (*cf.* *1·12·21), since a is distinct from x and y, x can be substituted for y in R'$(axyzt)$. Hence R'$(axxzt)$, which contradicts *1·61.

*1·63. *Proposition.*—Assuming β Hp R, then $\mathrm{R_{in}}$'(BCDt) implies $\mathrm{R_{in}}$'(DCBt). In symbols,

$$\vdash .\therefore \beta\,\mathrm{H}p\,\mathrm{R} \,.\supset: \mathrm{R_{in}}'(\mathrm{BCD}t) \,.\supset.\, \mathrm{R_{in}}'(\mathrm{DCB}t)$$

Proof.—*Cf.* *1·31·52.

*1·64. *Proposition.*—Assuming $(\alpha, \beta, \gamma, \delta)$ Hp R, then $\mathrm{R_{in}}$'(BCDt) and $\mathrm{R_{in}}$'(CDBt) are inconsistent. In symbols,

$$\vdash .\therefore (\alpha, \beta, \gamma, \delta)\,\mathrm{H}p\,\mathrm{R} \,.\supset: \mathrm{R_{in}}'(\mathrm{BCD}t) \,.\supset.\, \sim \mathrm{R_{in}}'(\mathrm{CDB}t)$$

Proof.—$\mathrm{R_{in}}$'(BCDt) implies (*cf.* *1·31) that a, x, y, z exist such that a is a common member of B, C, D, x is a member of B, y of C, z of D, and R'$(axyzt)$, and B, C, D are members of R'$(a???t)$. Hence (*cf.* *1·61) a, x, y, z are all distinct. Similarly also if $\mathrm{R_{in}}$'(CDBt), then a', x', y', z' exist with similar properties, viz., x' a member of B, &c., except that R'$(a'y'z'x't)$. Hence (*cf.* *1·23·62) a and a' are identical. Thus R'$(axyzt)$ and R'$(ay'z'x't)$. But (*cf.* *1·21) x can be substituted for x', y for y', and z for z'. Hence R'$(axyzt)$ and R'$(ayzxt)$. But this contradicts γ Hp R.

*1·65. *Proposition.*—Assuming (intpnt, α, β, γ, δ) Hp R, the classes R'$(;\cdots t)$ and R'$(;;;;t)$ are identical. In symbols,

$$\vdash : (\mathrm{intpnt}, \alpha, \beta, \gamma, \delta)\,\mathrm{H}p\,\mathrm{R} \,.\supset.\, \mathrm{R}'(;\cdots t) = \mathrm{R}'(;;;;t)$$

Proof.—If x is a member of $R^{\backprime}(\cdot;;;t)$, y exists such that x is a member of $R^{\backprime}(y;;;t)$; also (*cf.* *1·61) x and y are distinct. Hence (*cf.* *1·21) P exists such that it is a member of $R^{\backprime}(y???t)$, and x is a member of it. Hence (*cf.* *1·41) P is a member of $R^{\backprime}(x???t)$, and hence (*cf.* *1·21) y is a member of $R^{\backprime}(x;;;t)$. Hence x is a member of $R^{\backprime}(;\cdots t)$.

*1·71. *Proposition.*—Assuming $\alpha\,\mathrm{H}p\,\mathrm{R}$, every interpoint possesses at least two members. In symbols,

$$\vdash .\because \alpha\,\mathrm{H}p\,\mathrm{R}\,.\supset : \mathrm{A}\,\epsilon\,\mathrm{intpnt}_{\mathrm{R}t}\,.\supset .\,\mathrm{N}c^{\backprime}\mathrm{A} \geqq 2$$

Proof.—*Cf.* *1·13·21·22·51.

*1·72. *Proposition.*—Assuming (intpnt, α, β, γ, δ) Hp R, then on every objective real there exist at least three interpoints. In symbols,

$$\vdash .\because (\mathrm{intpnt}, \alpha, \beta, \gamma, \delta)\,\mathrm{H}p\,\mathrm{R}\,.\supset : a\,\epsilon\,\mathrm{R}^{\backprime}(;;;;t)\,.\supset .\,\mathrm{N}c^{\backprime}\mathrm{R}^{\backprime}(a???t) \geqq 3$$

Proof.—*Cf.* *1·21·31·62·65.

*1·73. *Proposition.*—Assuming (intpnt, α, β, γ, δ) Hp R, then, if there are any objective reals, the interpoints are not all on any one objective real. In symbols,

$$\vdash .\because (\mathrm{intpnt}, \alpha, \beta, \gamma, \delta)\,\mathrm{H}p\,\mathrm{R}\,.\supset : \exists\,!\,\mathrm{R}^{\backprime}(;;;;t)\,.\supset .\,\exists\,!\{\mathrm{intpnt}_{\mathrm{R}t} - \mathrm{R}^{\backprime}(a???t)\}$$

Proof.—*Cf.* *1·42·71·72.

(iii) CONCEPT IV.

*2. This concept bifurcates into two alternate forms, namely IVA. and IVB. Concept IVB. is related to IVA. just as Concept II. is related to the classical concept. Thus Concept IVA. is dualistic, and Concept IVB. is the monistic variant of it. Both concepts can initially be considered together as Concept IV. In Concept IV. the essential relation (R) is pentadic, one of the terms being an instant of time. $R^{\backprime}(abcdt)$ can be read as a intersects b, c and d, in the order bcd at the instant t. The class of those entities, appearing among the first four terms in any instance of the relation holding, is called the class (O) of "linear objective reals." The remaining class of objective reals, required for Concept IVA., is called the class of "particles."

The geometrical points of this concept are simply the interpoints of R, as defined above (*cf.* *1). During the consideration of this concept they will be called *points.* The further definitions, beyond those of *1, required for a concise statement of the geometrical axioms are almost exactly those of Concept I., with the R_{in} of this Concept IV. written for the R of Concept I., and modified by the mention of t, as in Concept III. This mention of t can be managed in a similar (though not identical) way to that in Concept III. by writing

*2·01. $$\mathrm{R}_t{}^{\backprime}(\mathrm{ABC}) = \mathrm{R}_{\mathrm{in}}{}^{\backprime}(\mathrm{AB}Ct) \qquad \mathrm{Df}$$

Then the definitions of Concept I. will be assumed to apply to R_t. For example, the *punctual line joining the points* A *and* B is the class of points which is the logical sum of $R_t{}^{\backprime}(;AB)$ and $R_t{}^{\backprime}(A;B)$ and $R_t{}^{\backprime}(AB;)$ together with A and B themselves. Its symbol is $R_t{}^{\backprime}\overline{AB}$. The definition in symbols is

$$R_t{}^{\backprime}\overline{AB} = R_t{}^{\backprime}(;AB) \cup R_t{}^{\backprime}(A;B) \cup R_t{}^{\backprime}(AB;) \cup \iota{}^{\backprime}A \cup \iota{}^{\backprime}B \qquad \text{Df}$$

It will follow (*cf.* *1·23·31) from the axioms that a punctual line is the class of those points with some member of O as sole common member. The other definitions can be managed in like manner, only in the symbolism a suffix to a suffix will be avoided by writing $\Delta_{Rt}{}^{\backprime}(ABC)$, and so on, instead of $\Delta_{R_t}{}^{\backprime}(ABC)$, and so on.

The Axioms.—The earlier axioms have to be modified from those of Concept I., but the later axioms are simply those of Concept I. with the R of that concept replaced by the R_t of Concept IV.

$$\text{I Hp R} . = : t \in T . \supset_t . O \subset R{}^{\backprime}(;;;;t) \qquad \text{Df}$$

$$\text{II Hp R} . = . \exists ! R{}^{\backprime}(\cdots\cdot;) \; [i.e., \exists ! T] \qquad \text{Df}$$

$$\text{III Hp R} . = . \alpha \text{ Hp R} \qquad \text{Df} \quad (cf. *1{\cdot}51)$$

$$\text{IV Hp R} . = . \beta \text{ Hp R} \qquad \text{Df} \quad (cf. *1{\cdot}52)$$

$$\text{V Hp R} . = . \gamma \text{ Hp R} \qquad \text{Df} \quad (cf. *1{\cdot}53)$$

$$\text{VI Hp R} . = . \delta \text{ Hp R} \qquad \text{Df} \quad (cf. *1{\cdot}54)$$

$$\text{VII Hp R} . = . \text{intpnt Hp R} \qquad \text{Df} \quad (cf. *1{\cdot}41)$$

$$\text{VIII Hp R} . = .\therefore (A, B, C) .\therefore A, B, C \in R_t{}^{\backprime}(;;;) . A \neq B . A \neq C . B \neq C . \exists !(A \cap B \cap C) . \supset : R_t{}^{\backprime}(ABC) . \vee . R_t{}^{\backprime}(BCA) . \vee . R_t{}^{\backprime}(CAB) \qquad \text{Df}$$

$$\text{IX Hp R} . = : (A, B) : A, B \in R_t{}^{\backprime}(;;;) . A \neq B . \supset . \exists ! R{}^{\backprime}(AB;) \qquad \text{Df}$$

$$\text{X Hp R} . = : (A, B, C, D, E) : \Delta_{Rt}{}^{\backprime}(ABC) . R_t{}^{\backprime}(BCD) . R_t{}^{\backprime}(CEA) . \supset . \exists ! \{R_t{}^{\backprime}\overline{DE} \cap R_t{}^{\backprime}(A;B)\} \qquad \text{Df}$$

$$\text{XI Hp R} . = : t \in T . \supset_t . (\exists p, D) . p \in \text{ple}_{Rt} . D \in R_t{}^{\backprime}(;;;) - p \qquad \text{Df}$$

$$\text{XII Hp R} . = . (\exists A, B, C, D) . R_t{}^{\backprime}(;;;) \subset \Pi_{Rt}(ABCD) \qquad \text{Df}$$

$$\text{XIII Hp R} . = . \text{the axiom of continuity, } cf. \text{ XII Hp R of Concept I.} \qquad \text{Df}$$

$$\text{XIV Hp R} . = .\therefore \alpha \in \text{ple}_{Rt} . a \in \text{lin}_{Rt} \cap \text{cls}{}^{\backprime}\alpha . \supset_{a,\alpha} : (\exists C) : C \in \alpha : l, l' \in \text{lin}_{Rt} \cap \text{cls}{}^{\backprime}\alpha . C \in l \cap l' . l \cap a = \Lambda . l' \cap a = \Lambda . \supset_{l,l'} . l = l' \qquad \text{Df}$$

Note that only I Hp R and XI Hp R require the hypothesis $t \in T$; in all the other axioms there is a hypothesis which can only be true when $t \in T$. For the purpose of comparison with the axioms of Concept I., the following propositions are required:—

$*2{\cdot}11.\ \vdash\therefore (\mathrm{III, IV, V, VI, VII})\,\mathrm{H}p\,\mathrm{R}\,.\supset: t\,\epsilon\,\mathrm{T}\,.\supset.\,\mathrm{N}c\text{‘}\mathrm{R}_t\text{‘}(;;;) \geqq 2$

Proof.—Cf. *1·72.

$*2{\cdot}21.\ \vdash\therefore \mathrm{IV}\,\mathrm{H}p\,\mathrm{R}\,.\supset: \mathrm{R}_t\text{‘}(\mathrm{ABC})\,.\supset.\,\mathrm{R}_t\text{‘}(\mathrm{CBA})$

Proof.—Cf. *1·63.

$*2{\cdot}22.\ \vdash\therefore (\mathrm{III, IV, V, VI})\,\mathrm{H}p\,\mathrm{R}\,.\supset: \mathrm{R}_t\text{‘}(\mathrm{ABC})\,.\supset.\sim\mathrm{R}_t\text{‘}(\mathrm{BCA})$

Proof.—Cf. *1·64.

$*2{\cdot}23.\ \vdash\therefore (\mathrm{III, IV, V, VI})\,\mathrm{H}p\,\mathrm{R}\,.\supset: \mathrm{R}_t\text{‘}(\mathrm{ABC})\,.\supset.\,\mathrm{A}\neq\mathrm{C}$

Proof.—Cf. *1·62.

*2·31. *Proposition.*—Assuming (VII, IX) Hp R, if A and B are two distinct points at the time t, they possess one, and only one, common member. In symbols,

$$\vdash\therefore (\mathrm{VII, IX})\,\mathrm{H}p\,\mathrm{R}\,.\supset: \mathrm{A, B}\,\epsilon\,\mathrm{R}_t\text{‘}(;;;)\,.\,\mathrm{A}\neq\mathrm{B}\,.\supset.\,\mathrm{A}\cap\mathrm{B}\,\epsilon\,1$$

Proof.—Cf. *1·31·42 and (VII, IX) Hp R.

*2·32. *Proposition.*—Assuming (VII, VIII, IX) Hp R, a line at any instant t (*i.e.*, a member of $\mathrm{lin}_{\mathrm{R}t}$) is the complete class of points (interpoints) possessing some linear objective real. In symbols,

$$\vdash: (\mathrm{VII, VIII, IX})\,\mathrm{H}p\,\mathrm{R}\,.\supset.\,\mathrm{lin}_{\mathrm{R}t} = \acute{p}\,[(\exists\alpha)\,.\,\alpha\,\epsilon\,\mathrm{R}\text{‘}(;;;;t)\,.\,p = \acute{\mathrm{A}}\,\{\mathrm{A}\,\epsilon\,\mathrm{R}_t\text{‘}(;;;)\,.\,\alpha\,\epsilon\,\mathrm{A}\}]$$

Proof.—Cf. VIII Hp R and *2·31.

Propositions *2·31·32 effect the identification of the punctual line, as defined above, and the class of points on some linear objective real. Thus a straight line considered as an entity with parts is a punctual line, and considered as a simple unit is a linear objective real.

*2·33. *Proposition.*—Assuming (VII, VIII, IX) Hp R, if C and D are two points in the punctual line $\mathrm{R}_t\text{‘}\overline{\mathrm{AB}}$, then A is a point in the punctual line $\mathrm{R}_t\text{‘}\overline{\mathrm{CD}}$. In symbols,

$$\vdash\therefore (\mathrm{VII, VIII, IX})\,\mathrm{H}p\,\mathrm{R}\,.\supset: \mathrm{C, D}\,\epsilon\,\mathrm{R}_t\text{‘}\overline{\mathrm{AB}}\,.\,\mathrm{C}\neq\mathrm{D}\,.\supset.\,\mathrm{A}\,\epsilon\,\mathrm{R}_t\text{‘}\overline{\mathrm{CD}}$$

Proof.—Cf. *2·32.

$*2{\cdot}41.\ \vdash\therefore (\mathrm{III{-}IX})\,\mathrm{H}p\,\mathrm{R}\,.\supset: t\,\epsilon\,\mathrm{T}\,.\supset.\,(\exists\,\mathrm{A, B, C})\,.\,\triangle_{\mathrm{R}t}\text{‘}(\mathrm{ABC})$

Proof.—Cf. *1·72·73.*2·32.

*2·5. *Proposition.*—Assuming (III–XIV) Hp R of Concept IV., then all the axioms of Concept I. hold when the R_t of Concept IV. is substituted for the R of Concept I., and t is a member of T.

Proof.—Cf. *2·11·21·22·23·33·41 and (IX–XIV) Hp R (of Concept IV.) and (I–XII) Hp R of Concept I.

It will be noticed that I Hp R (of Concept IV.) is not required in the above

comparison. It does not belong to the purely geometrical side of the concept, but is a necessary part of the "physical" ideas. II H*p* R (of Concept IV.), though it does not occur explicitly in the above comparison, is required to give the geometry "existence." Thus the geometry of Concept IV. requires thirteen axioms.

For the purpose of the transition to projective geometry (*cf.* VEBLEN, *loc. cit.*), it is now unnecessary to conceive a new class of "projective points." The points already on hand are exactly the entities required. All that is necessary is to define the class of those linear objective reals (*cf.* XIV H*p* R), coplanar with any given linear objective real and not intersecting it, as *the point at infinity* on that objective real. Then with these new points at infinity, and the old points, the complete set of "projective points" is obtained.

The Extraneous Relation.—For the purpose of the definition of motion, one extraneous tetradic relation is required, exactly as in Concept III. Also the same hypotheses must hold respecting it. The three mutually rectangular and intersecting punctual lines, thus indicated at each instant, are to be taken as the "kinetic axes," and all motion measured by reference to them. A given set of kinetic axes does not, in general, correspond to the same three linear objective reals at different instants of time.

Matter.—It is necessary to assume that the points in this concept disintegrate, and do not, in general, persist from instant to instant. For otherwise the only continuous motion possible would be representable by linear transformations of coordinates; and it seems unlikely that sense-perceptions could be explained by such a restricted type of motions. We have therefore to consider what, in this concept, can represent the permanence of matter. A "corpuscle," as we may call it, may be conceived to be a volume with some special property in respect to the linear objective reals "passing through" it. This is the procedure adopted in Concept V.; and the methods of overcoming the obvious difficulties which suggest themselves will be considered in detail there. It is sufficient here to notice that, in this Concept IV., the special property of the volume must relate merely to the motion of the objective reals. For the only alternative is to make the property consist of the permanence of the points within the volume. But then the difficulty of permanent collineations, mentioned above, recurs. To find a special property of motion, we require a kinematical science for linear objective reals in this concept analogous to the kinematical parts of hydrodynamics. In the absence at the present time of such a science, we proceed to other alternatives.

Concept IVA.—Conceive a class of *particles*, each particle being associated at each instant with some point, but not necessarily each point with some particle. Then the particles represent the "matter" which "occupies" space. Laws of motion must then be stated (i) for the particles and (ii) for the linear objective reals. Also the motion of the particles may be conceived to be influenced by that of the linear objective reals, and *vice versâ*. The endeavour to state such laws appears to reduce

itself to rewriting with appropriate changes a chapter of any modern treatise of electricity and magnetism. It would seem necessary to subdivide the class of particles into "positive" and "negative" particles, a charged volume containing an excess of one type. The conception of an ether conveying lines of force is replaced by the class of the linear objective reals. The details can be managed much as in the analogous case of Concept V., considered later. An indefinite number of extraneous relations are required to "locate" the particles, just as in Concept I. This concept (as thus developed with "particles") is not completely a "linear" concept. It is a hybrid between the "linear" and "punctual" concepts. In its dualism it is not superior to the classical concept. But, in possessing moving linear objective reals as well as moving particles, it is richer in physical ideas.

Concept IVB.—In this concept, just as in Concept II., each triadic extraneous relation of Concept IVA. between an instant of time, a particle, and a point is replaced by a dyadic extraneous relation between a point and an instant of time.

PART IV.—(i) THE THEORY OF DIMENSIONS.

*3. Concept V. depends upon a treatment of the theory of dimensions different from that which at present obtains. The theory here developed is relevant to any definite property which (1) is a property of classes only, and (2) is only a property of some classes. It will be clearer, and no longer, to explain the theory in its full generality, and in Concept V. to make the special application required.

This general theory of dimensions may, perhaps, have a range of importance greater than that which is assigned to it in the sequel. In *10 a set of hypotheses are given respecting the property ϕ; and when these are true of ϕ, the propositions and definitions of *3 to *8 acquire importance and emerge from triviality, also in this case further deductions of propositions can be made. The Concept V. to which this theory is applied is explained in the definitions of *20 and the axioms of *22. In this Concept V. a special property ϕ is taken, which is termed "Homaloty" (*cf.* *20·11·12), and (*cf.* *22) in the axioms a relation R is considered such that "homaloty," defined in respect to R, has the properties of the axioms in *10.

*3·01. *Definition.*—If $\phi!x$ is some proposition involving the entity x, which may be varied, so that $\phi!x$ and $\phi!y$ make the same statement (ϕ) about x and y respectively, then any entity z, for which $\phi!z$ is true, is said to *possess the property* ϕ.

*3·02. *Definition.*—A *ϕ-class* is a class with the property ϕ, that is to say, if u is a ϕ-class, then $\phi!u$ is true.

*3·11. *Definition.*—The *ϕ-region* is the logical sum of all classes which possess the property ϕ. The symbol O_ϕ will denote the ϕ-region. The symbolic definition is

$$O_\phi = \cup`\hat{u}\{\phi!u\} \qquad \text{Df}$$

*3·12. The *common ϕ-subregion for u* is that class which is the common subclass

of all ϕ-classes with the class u as a subclass. The symbol $\mathrm{cm}_\phi`u$ will denote the common ϕ-subregion for u. The symbolic definition is

$$\mathrm{cm}_\phi`u = \cap`\hat{v}\{\phi!v \,.\, u \in \mathrm{cls}`v\} \qquad \mathrm{Df}$$

Note.—If no class v, with the property ϕ and containing u as a subclass, exists, then $\mathrm{cm}_\phi`u$ will be the class of all entities. But if a class $v$ exists which has the property $\phi$ and contains $u$, then $\mathrm{cm}_\phi`u$ is a subclass of O_ϕ. In the sequel it will be found that this latter is the only relevant case for our purposes.

Elucidatory Note.—Assuming our ordinary geometrical ideas, let the property of the "flatness" of a class of straight lines be defined thus: A class of straight lines is flat, *either*, when it is a necessary and sufficient condition for membership that a straight line meets two members of the class, not at their point of meeting, *or*, when the class is a unit class with one line as its sole member. Thus a plane (as a line-locus) is flat, a three-dimensional space (as a line-locus) is flat, and so on. Now let the property ϕ in the above definition be the property of flatness. If then u is a class consisting only of two straight lines, the common ϕ-subregion for u is either a three-dimensional space or a plane, according as the two lines are not, or are, coplanar. Also in a space of higher dimensions than three, if u be a class consisting of three straight lines, the common ϕ-subregion for u may be either (1) a plane, or (2) a three-dimensional space, or (3) a four-dimensional space, or (4) a five-dimensional space, according to the circumstances of the lines. It will be noticed that, in the application of this theory of the common ϕ-subregion to the particular case of geometrical flatness, the common ϕ-subregion of any class of lines is itself flat. But this is not, in general, the case when any property not flatness is considered. It is this peculiar property of flatness which has masked the importance in geometry of the theory of common ϕ-subregions.

*3·121. *Definition.*—Two classes u and v have *ϕ-equivalence* if $\mathrm{cm}_\phi`u = \mathrm{cm}_\phi`v$. The class of those classes (including u itself as a member), which have ϕ-equivalence with u, is denoted by $\mathrm{equiv}_\phi`u$. The symbolic definition is

$$\mathrm{equiv}_\phi`u = \hat{v}\,(\mathrm{cm}_\phi`v = \mathrm{cm}_\phi`u) \qquad \mathrm{Df}$$

*3·13. *Definition.*—A class u (not the null class) is *ϕ-prime*, when, if v be any proper part (part, not the whole) of u, v is *not* ϕ-equivalent to u. The class of those classes which are ϕ-prime will be denoted by the symbol prm_ϕ. The symbolic definition is

$$\mathrm{prm}_\phi = \hat{u}\{\exists!u : v \subset u \,.\, \exists!(u - v) \,.\, \supset_v .\, \mathrm{cm}_\phi`v \neq \mathrm{cm}_\phi`u\} \qquad \mathrm{Df}$$

Elucidatory Note.—With the assumptions of the elucidatory note on $\mathrm{cm}_\phi`u$, it is at once obvious that two straight lines form a $\phi$-prime (where $\phi$ is flatness) class, whether they are or are not coplanar. But if $u$ consist of three straight lines, (1) $u$ is not $\phi$-prime if $\mathrm{cm}_\phi`u$ is a plane, (2) u is not, in general, ϕ-prime if $\mathrm{cm}_\phi`u$ is a space of three dimensions, but u is (in this case) ϕ-prime if the three lines are concurrent,

(3) u is ϕ-prime if $\text{cm}_\phi{}^\backprime u$ is of four dimensions, (4) u is ϕ-prime if $\text{cm}_\phi{}^\backprime u$ is of five dimensions.

*3·21. The *ϕ-dimension number* (or the *ϕ-dimensions*) of a class u is the greatest of the cardinal numbers of all classes (including possibly u itself) which are both ϕ-equivalent to u and ϕ-prime. The ϕ-dimension number of u will be denoted by $\text{dim}_\phi{}^\backprime u$. The symbolic definition is

$$\text{dim}_\phi{}^\backprime u \,.=:(\iota\alpha):\alpha\,\epsilon\,\text{N}c^{\backprime\backprime}(\text{prm}_\phi \cap \text{equiv}_\phi{}^\backprime u):\rho\,\epsilon\,\text{N}c^{\backprime\backprime}(\text{prm}_\phi \cap \text{equiv}_\phi{}^\backprime u)\,.\supset_\rho.\,\rho \leqq \alpha \qquad \text{Df}$$

Elucidatory Note.—With the assumptions of the previous elucidatory notes (where ϕ is flatness), we see that those ϕ-prime classes, the common ϕ-subregions for which are spaces of three dimensions (as ordinarily understood), are all pairs of non-intersecting lines and all trios of concurrent non-coplanar lines; also no class of four lines in such a space can be prime. Thus three is the greatest cardinal number of any ϕ-prime class of lines for which the common ϕ-subregion is such a space. Hence, according to the above definition, three is the ϕ-dimension number of the space.

*3·22. *Definition.*—A class u is *ϕ-axial* when (1) it is ϕ-prime and (2) its cardinal number is equal to its ϕ-dimensions. The class of ϕ-axial classes is denoted by the symbol ax_ϕ. The symbolic definition is

$$\text{ax}_\phi = \hat{u}\{u\,\epsilon\,\text{prm}_\phi \cap \text{dim}_\phi{}^\backprime u\} \qquad \text{Df}$$

Elucidatory Note.—With the assumptions of the previous elucidatory notes (where ϕ is flatness), we see that two coplanar lines form a ϕ-axial class, and so also do three concurrent non-coplanar lines.

*3·23. *Definition.*—A class u is *ϕ-maximal* when (1) all those of its subclasses (possibly including u itself), which are both ϕ-prime and ϕ-equivalent to u, are ϕ-axial, and (2) there are such subclasses. The class of ϕ-maximal classes will be denoted by mx_ϕ. The symbolic definition is

$$\text{mx}_\phi = \hat{u}\{\exists\,!(\text{prm}_\phi \cap \text{equiv}_\phi{}^\backprime u \cap \text{cls}^\backprime u)\,.\,\text{prm}_\phi \cap \text{equiv}_\phi{}^\backprime u \cap \text{cls}^\backprime u \subset \text{ax}_\phi\} \qquad \text{Df}$$

Elucidatory Note.—Referring to the previous elucidatory notes (where ϕ is flatness), we see that any set of coplanar lines form a ϕ-maximal class; similarly any set of concurrent lines form a ϕ-maximal class.

*3·31. *Definition.*—The *ϕ-concurrence* of u and v, where u and v are classes, is that subclass of u (possibly u itself), such that any couple, formed by any member of it and any member of v, is ϕ-axial. The ϕ-concurrence of u with v is denoted by the symbol $\tilde{u}_\phi{}^\backprime v$. The definition in symbols is

$$\tilde{u}_\phi{}^\backprime v = \hat{x}\{x\,\epsilon\,u : y\,\epsilon\,v\,.\supset_y.\,\iota^\backprime x \cup \iota^\backprime y\,\epsilon\,\text{ax}_\phi\} \qquad \text{Df}$$

The ϕ-concurrence of the ϕ-region (O_ϕ) with any class v will be written $\tilde{\text{O}}_\phi{}^\backprime v$ instead of $\tilde{\text{O}}_{\phi_\phi}{}^\backprime v$.

Elucidatory Note.—Referring to the previous elucidatory notes (when ϕ is flatness),

we see that, when u and v are classes of straight lines, $\tilde{u}_\phi\text{‘}v$ (*i.e.*, the ϕ-concurrence of u with v) is that complete set of lines of u which is such that any member of it is coplanar with every member of v.

*3·32. *Definition.*—A class u is a *self-ϕ-concurrence* if the ϕ-concurrence of u with itself is the whole class u. The class of those classes which are self-ϕ-concurrences will be denoted by conc_ϕ. The symbolic definition is

$$\text{conc}_\phi = \hat{u}\,\{u = \tilde{u}_\phi\text{‘}u\} \qquad \text{Df}$$

*3·33. *Definition.*—x will be said to be *ϕ-concurrent with y*, if the class composed of x and y only (*i.e.*, the class $\iota\text{‘}x \cup \iota\text{‘}y$) is ϕ-axial.

*3·41. *Definition.*—A *ϕ-plane* is a class u such that there exists a class v, which (1) is ϕ-axial, (2) is composed of two members only, and (3) is such that u is the class $\text{cm}_\phi\text{‘}v$. The class of those classes which are ϕ-planes is denoted by ple_ϕ. The symbolic definition is

$$\text{ple}_\phi = \hat{u}\,\{(\exists v)\,.\,v \in 2 \cap \text{ax}_\phi\,.\,u = \text{cm}_\phi\text{‘}v\} \qquad \text{Df}$$

Note.—It requires an axiom to establish that a ϕ-plane is a self-ϕ-concurrence (*cf.* *16·11).

*3·42. *Definition.*—A class u is a *ϕ-point*, if there exists a class v, which (1) is ϕ-axial, (2) is composed of three members only, and (3) is such that u is the ϕ-concurrence of the ϕ-region with v. The class of those classes which are ϕ-points is denoted by the symbol pnt_ϕ. The symbolic definition is

$$\text{pnt}_\phi = \hat{u}\,\{(\exists v)\,.\,v \in 3 \cap \text{ax}_\phi\,.\,u = \tilde{\text{O}}_\phi\text{‘}v\} \qquad \text{Df}$$

Note.—It requires axioms to establish that a ϕ-point is ϕ-maximal and is a self-ϕ-concurrence (*cf.* *14·11·12). Also note that this definition does not apply unless the number of dimensions of O_ϕ is at least three, but then applies *unchanged* however great this number may be.

Elucidatory Note.—Referring to the previous elucidatory notes (where ϕ is flatness), we see that a ϕ-point now becomes simply that class of straight lines concurrent at a point. The analogy with KLEIN's "ideal," or "projective," points is obvious. Only when the present theory is applied, it will be found that the original "descriptive" point has entirely vanished.

*3·43. *Definition.*—A class is *ϕ-coplanar* if there exists a ϕ-plane of which it is a subclass. The symbol $\text{cople}_\phi!u$ denotes that the class u is ϕ-coplanar. The definition in symbols is

$$\text{cople}_\phi!u\,.=.\,(\exists p)\,.\,p \in \text{ple}_\phi\,.\,u \in \text{cls}\text{‘}p \qquad \text{Df}$$

*3·44. *Definition.*—A class is *ϕ-copunctual* if there exists a ϕ-point of which it is a subclass. The symbol $\text{copnt}_\phi!u$ denotes that the class u is ϕ-copunctual. The definition in symbols is

$$\text{copnt}!u\,.=.\,(\exists \text{P})\,.\,\text{P} \in \text{pnt}_\phi\,.\,u \in \text{cls}\text{‘P} \qquad \text{Df}$$

General Deductions Concerning Dimensions.

A large chapter of interesting propositions concerning the entities defined above in *3 can be compiled. The following are chosen as being directly wanted in the subsequent investigations :—

*4. *On Common ϕ-subregions.*

*4·21. *Proposition.*—If v is a subclass of u, then $\mathrm{cm}_\phi{}'v$ is a subclass of $\mathrm{cm}_\phi{}'u$. In symbols,

$$\vdash : v \subset u \,.\supset.\, \mathrm{cm}_\phi{}'v \subset \mathrm{cm}_\phi{}'u$$

Proof.—*Cf.* *3·12.

*4·25. *Proposition.*—A class u is itself a subclass of $\mathrm{cm}_\phi{}'u$. In symbols

$$\vdash .\, u \subset \mathrm{cm}_\phi{}'u$$

Proof.—*Cf.* *3·12.

*4·27. *Proposition.*—If u is a class with the property ϕ, then u is identical with $\mathrm{cm}_\phi{}'u$. In symbols,

$$\vdash : \phi ! u \,.\supset.\, u = \mathrm{cm}_\phi{}'u$$

Proof.—*Cf.* *3·12.

*4·28. *Proposition.*—If there exist two classes, both with the property ϕ, which possess no common member, then $\mathrm{cm}_\phi{}'\Lambda$ is itself the null class (Λ). In symbols,

$$\vdash : (\exists\, u, v) \,.\, u \cap v = \Lambda \,.\, \phi ! u \,.\, \phi ! v \,.\supset.\, \mathrm{cm}_\phi{}'\Lambda = \Lambda$$

Proof.—Note that $\mathrm{cm}_\phi{}'\Lambda$ is the common part of all ϕ-classes.

Corollary.—If x and y exist such that they are distinct, and the two unit classes with them as members respectively each have the property ϕ, then $\mathrm{cm}_\phi{}'\Lambda$ is Λ.

Note that when *4·28 is appealed to, it will be this corollary which is directly used.

*4·31. *Proposition.*—The common ϕ-subregion for the common ϕ-subregion for u is the common ϕ-subregion for u. In symbols,

$$\vdash .\, \mathrm{cm}_\phi{}'\mathrm{cm}_\phi{}'u = \mathrm{cm}_\phi{}'u$$

Proof.—For $\mathrm{cm}_\phi{}'u$ is contained in every ϕ-class containing u. Hence (*cf.* *3·12) $\mathrm{cm}_\phi{}'\mathrm{cm}_\phi{}'u$ is contained in $\mathrm{cm}_\phi{}'u$. Also (*cf.* *4·25·21) $\mathrm{cm}_\phi{}'u$ is contained in $\mathrm{cm}_\phi{}'\mathrm{cm}_\phi{}'u$.

*4·32. *Proposition.*—If u and v are ϕ-equivalent, and w is any class, then the common ϕ-subregion for the logical sum of u and w is identical with the common ϕ-subregion for the logical sum of v and w. In symbols,

$$\vdash : \mathrm{cm}_\phi{}'u = \mathrm{cm}_\phi{}'v \,.\supset.\, \mathrm{cm}_\phi{}'(u \cup w) = \mathrm{cm}_\phi{}'(v \cup w)$$

Proof.—For (*cf.* *4·21) $\mathrm{cm}_\phi{}'v$ is contained in $\mathrm{cm}_\phi{}'(v \cup w)$, and hence (*hypothesis*) $\mathrm{cm}_\phi{}'u$ is contained in $\mathrm{cm}_\phi{}'(v \cup w)$, and hence (*cf.* *4·25) $u \cup w$ is contained in $\mathrm{cm}_\phi{}'(v \cup w)$, and hence (*cf.* *4·21) $\mathrm{cm}_\phi{}'(u \cup w)$ is contained in $\mathrm{cm}_\phi{}'\mathrm{cm}_\phi{}'(v \cup w)$, and

hence (*cf.* *4·31) cm_ϕ‘$(u \cup w)$ is contained in cm_ϕ‘$(v \cup w)$. Then interchanging u and v, and combining the two results, the proposition follows.

The following propositions are not cited subsequently, so their verbal enunciations are omitted :—

*4·41. $\vdash . \mathrm{cm}_\phi`(\mathrm{cm}_\phi`u \cap \mathrm{cm}_\phi`v) = \mathrm{cm}_\phi`u \cap \mathrm{cm}_\phi`v$

*4·42. $\vdash . \cap`\mathrm{cm}_\phi``p = \mathrm{cm}_\phi` \cap `\mathrm{cm}_\phi``p$

*4·43. $\vdash . \mathrm{cm}_\phi`(\mathrm{cm}_\phi`u \cup \mathrm{cm}_\phi`v) = \mathrm{cm}_\phi`(u \cup v)$

*4·44. $\vdash . \mathrm{cm}_\phi` \cup `\mathrm{cm}_\phi``p = \mathrm{cm}_\phi` \cup `p$

*5. *On ϕ-Primes.*

*5·23. *Proposition.*—If u is a ϕ-prime, and v is a subclass of u, and is not the null class, then v is a ϕ-prime. In symbols,

$$\vdash : u \in \mathrm{prm}_\phi \,.\, v \in \mathrm{cls}`u \,.\, \exists ! v \,.\supset .\, v \in \mathrm{prm}_\phi$$

Proof.—For if w be any subclass (not the null class) of v, then (*cf.* *3·13) cm_ϕ‘$(u - w)$ is not cm_ϕ‘u. But $(u - w)$ can be written $\{(v - w) \cup (u - v)\}$, and u can be written $\{v \cup (u - v)\}$. Hence cm_ϕ‘$\{(v - w) \cup (u - v)\}$ is not cm_ϕ‘$\{v \cup (u - v)\}$. Hence (*cf.* *4·32) cm_ϕ‘$(v - w)$ is not cm_ϕ‘v. Hence (*cf.* *3·13) v is a ϕ-prime.

Note.—This theorem, together with *4·31·32, is the foundation of the whole theory. It is remarkable that it requires no axiom concerning ϕ. The companion theorem (*cf.* *12·42), with ax_ϕ substituted for prm_ϕ, requires axioms respecting ϕ.

*5·231. *Proposition.*—Necessary and sufficient conditions, that a class u may be ϕ-prime, are: (1) u is not the null class, and (2) if x be any member of u, then cm_ϕ‘$(u - \iota`x)$ is not cm_ϕ‘u. In symbols,

$$\vdash .\because \exists ! u : x \in u \,.\supset_x .\, \mathrm{cm}_\phi`(u - \iota`x) \neq \mathrm{cm}_\phi`u : \equiv .\, u \in \mathrm{prm}_\phi$$

Proof.—*Cf.* *3·13 and *4·21.

*5·233. *Proposition.*—If $\mathrm{cm}_\phi`\Lambda = \Lambda$, then every unit class is a ϕ-prime. In symbols,

$$\vdash : \mathrm{cm}_\phi`\Lambda = \Lambda \,.\supset .\, 1 \subset \mathrm{prm}_\phi$$

Proof.—*Cf.* *3·13 and *4·25.

*5·235. *Proposition.*—If x and y are distinct, and the unit classes $\iota`x$ and $\iota`y$ have the property ϕ, then the class, which is the couple composed of x and y, is a ϕ-prime. In symbols,

$$\vdash : x \neq y \,.\, \phi ! \iota`x \,.\, \phi ! \iota`y \,.\supset .\, \iota`x \cup \iota`y \in \mathrm{prm}_\phi$$

Proof.—*Cf.* *3·13 and *4·25·27.

*6. *On ϕ-Dimensions and ϕ-Axial Classes.*

*6·23. *Proposition.*—The ϕ-dimension of u, if there is such an entity, is a cardinal number not zero. In symbols,

$$\vdash : (\mathrm{E}!)(\mathrm{dim}_\phi`u) \,.\supset .\, \mathrm{dim}_\phi`u \in \mathrm{N}c - \iota`0$$

Proof.—*Cf.* *3·21.

*6·25. *Proposition.*—If v is a ϕ-prime and has a ϕ-dimension number, then the cardinal number of v is less than, or equal to, $\mathrm{dim}_\phi \text{‘} v$. In symbols,

$$\vdash : (\mathrm{E}x)(\mathrm{dim}_\phi \text{‘} v) \,.\, v \,\epsilon\, \mathrm{prm}_\phi \,.\, \supset \,.\, \mathrm{N}c\text{‘}v \leqq \mathrm{dim}_\phi \text{‘} v$$

Proof.—*Cf.* *3·21.

*6·26. *Proposition.*—If v is ϕ-axial and is ϕ-equivalent to u, then the cardinal number of v is equal to $\mathrm{dim}_\phi \text{‘} u$. In symbols,

$$\vdash : v \,\epsilon\, \mathrm{ax}_\phi \cap \mathrm{equiv}_\phi \text{‘} u \,.\, \supset \,.\, \mathrm{N}c\text{‘}v = \mathrm{dim}_\phi \text{‘} u$$

Proof.—*Cf.* *3·21·22.

*8. *On ϕ-Concurrences.*

*8·21. *Proposition.*—If u is contained in w, then the ϕ-concurrence of u with v is contained in the ϕ-concurrence of w with v. In symbols,

$$\vdash : u \subset w \,.\, \supset \,.\, \tilde{u}_\phi \text{‘} v \subset \tilde{w}_\phi \text{‘} v$$

Proof.—*Cf.* *3·31.

*8·22. *Proposition.*—If v is contained in w, then the ϕ-concurrence of u with w is contained in the ϕ-concurrence of u with v. In symbols,

$$\vdash : v \subset w \,.\, \supset \,.\, \tilde{u}_\phi \text{‘} w \subset \tilde{u}_\phi \text{‘} v$$

Proof.—*Cf.* *3·31.

*10. *Geometrical Properties.*—A property ϕ is called *geometrical* if it satisfies the five axioms (λ, μ, ν, π, ρ) $\mathrm{H}p\,\phi$ stated below. The axiom $\nu\,\mathrm{H}p\,\phi$ takes the special form for three dimensions. It is to be noticed that three dimensions is the lowest number for which a ϕ-point (*cf.* *3·42) can be defined. The reasoning can be applied to higher dimensions, only more elaborate inductions and an extra axiom are required. Other axioms and definitions are wanted to enable all the propositions of projective geometry to be proved. These will not be considered here as such an investigation would involve some repetition when we come to Concept V. The class O_ϕ is the class of straight lines of the geometry, conceived as simple unities. The class pnt_ϕ is the class of points, each point being a class of lines. The class ple_ϕ is the class of planes, each plane being a class of lines.

*10·1. $\lambda\,\mathrm{H}p\,\phi$ is the statement that O_ϕ *has the property* ϕ. In symbols,

$$\lambda\,\mathrm{H}p\,\phi \,.\, = \,.\, \phi \,!\, \mathrm{O}_\phi \qquad \mathrm{Df}$$

*10·2. $\mu\,\mathrm{H}p\,\phi$ is the statement that, *if x is any member of* O_ϕ, *the unit class* $\iota\text{‘}x$ *has the property* ϕ. In symbols,

$$\mu\,\mathrm{H}p\,\phi \,.\, = : x \,\epsilon\, \mathrm{O}_\phi \,.\, \supset_x \,.\, \phi \,!\, \iota\text{‘}x \qquad \mathrm{Df}$$

*10·3. $\nu\,\mathrm{H}p\,\phi$ is the statement that *the ϕ-dimension number of* O_ϕ *is three.* In symbols,

$$\nu\,\mathrm{H}p\,\phi \,.\, = \,.\, \mathrm{dim}_\phi \text{‘} \mathrm{O}_\phi = 3 \qquad \mathrm{Df}$$

*10·4. π Hp ϕ is the statement that, *if u is a subclass of* O_ϕ, *and v is ϕ-axial and contained in* $\text{cm}_\phi\text{‘}u$, *then there exists a class w (possibly the null class) such that the logical sum of v and w is ϕ-axial and ϕ-equivalent to u.* In symbols,

$$\pi\,\text{Hp}\,\phi\,.=:u\,\epsilon\,\text{cls‘O}_\phi\,.\,v\,\epsilon\,\text{ax}_\phi \cap \text{cls‘cm}_\phi\text{‘}u\,.\supset_{u,v}.\,(\exists w)\,.\,v \cup w\,\epsilon\,\text{ax}_\phi \cap \text{equiv}_\phi\text{‘}u \qquad \text{Df}$$

*10·5. ρ Hp ϕ is the statement that *if u and v are both ϕ-axial, and if they possess at least two members in common, then their logical sum is ϕ-maximal.* In symbols,

$$\rho\,\text{Hp}\,\phi\,.=:u, v\,\epsilon\,\text{ax}_\phi\,.\,\text{N}c\text{‘}(u \cap v) \geqq 2\,.\supset_{u,v}.\,u \cup v\,\epsilon\,\text{mx}_\phi \qquad \text{Df}$$

Elucidatory Note.—Referring to the previous elucidatory notes (where ϕ is flatness), we see that *10·4 in effect assumes that a line can always be added (1) to two concurrent lines to form a set of three concurrent non-coplanar lines, and (2) to one line in a plane to form a set of two concurrent lines in that plane. Also *10·5 assumes that, if two sets of three concurrent lines have two members in common, the four lines are concurrent.

Deductions from the Axioms.

*11. *Preliminary Propositions.*

*11·11. *Proposition.*—Assuming (λ, ν) Hp ϕ, O_ϕ has at least three members. In symbols,

$$\vdash:(\lambda, \nu)\,\text{Hp}\,\phi\,.\supset.\,\text{N}c\text{‘O}_\phi \geqq 3$$

Proof.—*Cf.* *4·27 and *10·1·3.

*11·12. *Proposition.*—Assuming (λ, μ, ν) Hp ϕ, $\text{cm}_\phi\text{‘}\Lambda$ is the null class (Λ). In symbols,

$$\vdash:(\lambda, \mu, \nu)\,\text{Hp}\,\phi\,.\supset.\,\text{cm}_\phi\text{‘}\Lambda = \Lambda$$

Proof.—*Cf.* *4·28 and *10·2 and *11·11.

*11·21. *Proposition.*—Assuming (λ, μ) Hp ϕ, all ϕ-prime classes with more than one member are contained in O_ϕ. In symbols,

$$\vdash.\therefore(\lambda, \mu)\,\text{Hp}\,\phi\,.\supset:v\,\epsilon\,\text{prm}_\phi\,.\,\text{N}c\text{‘}v > 1\,.\supset.\,v\,\epsilon\,\text{cls‘O}_\phi$$

Proof.—For if x is not a member of O_ϕ, then $\text{cm}_\phi\text{‘}\iota\text{‘}x$ is the class of all entities. Hence (*cf.* *4·21·27 and *10·1·2) the conclusion follows.

*12. *On ϕ-Axial Classes and ϕ-Dimensions.*

*12·11. *Proposition.*—Assuming (λ, μ, ν) Hp ϕ, every unit class whose single member belongs to O_ϕ is ϕ-axial. In symbols,

$$\vdash.\therefore(\lambda, \mu, \nu)\,\text{Hp}\,\phi\,.\supset:x\,\epsilon\,\text{O}_\phi\,.\supset_x.\,\iota\text{‘}x\,\epsilon\,\text{ax}_\phi$$

Proof.—*Cf.* *3·21·22 and *4·27 and *5·233 and *10·2.

*12·12. *Proposition.*—Assuming $(\lambda - \pi)$ Hp ϕ, every subclass of O_ϕ, not the null

class, has a set of ϕ-axes. (*Note.*—A class which is ϕ-axial and ϕ-equivalent to a class u is said to be *a set of ϕ-axes of u.*) In symbols,

$$\vdash .\because (\lambda - \pi) \operatorname{Hp} \phi . \supset : u \in \text{cls}`\text{O}_\phi . \exists ! u . \supset . \exists ! (\text{ax}_\phi \cap \text{equiv}_\phi`u)$$

Proof.—Since there is at least one member of u, there is (*cf.* *12·11) a ϕ-axial class contained in u. Hence (*cf.* *10·4) this class can be augmented so as to become a set of ϕ-axes of u.

*12·13. *Proposition.*—Assuming $(\lambda - \pi) \operatorname{Hp} \phi$, if u and v are subclasses of O_ϕ, and v is not the null class, and $\text{cm}_\phi`v$ is contained in $\text{cm}_\phi`u$, then there exist two subclasses of O_ϕ, w and w', say, such that w is a set of ϕ-axes of v, and $w \cup w'$ is a set of ϕ-axes of u. In symbols,

$$\vdash .\because (\lambda - \pi) \operatorname{Hp} \phi . \supset : u, v \in \text{cls}`\text{O}_\phi . \exists ! v . \text{cm}_\phi`v \subset \text{cm}_\phi`u . \supset .$$
$$(\exists w, w') . w \in \text{ax}_\phi \cap \text{equiv}_\phi`v . w \cup w' \in \text{ax}_\phi \cap \text{equiv}_\phi`u.$$

Proof.—*Cf.* *10·4 and *12·12.

*12·21. *Proposition.*—Assuming $(\lambda - \pi) \operatorname{Hp} \phi$, if u and v are subclasses of O_ϕ, and v is not the null class, and $\text{cm}_\phi`v$ is contained in $\text{cm}_\phi`u$, then the ϕ-dimension number of v is less than, or equal to, the ϕ-dimension number of u. In symbols,

$$\vdash .\because (\lambda - \pi) \operatorname{Hp} \phi . \supset : u, v \in \text{cls}`\text{O}_\phi . \exists ! v . \text{cm}_\phi`v \subset \text{cm}_\phi`u . \supset . \dim_\phi`v \leqq \dim_\phi`u.$$

Proof.—From *6·26 and *12·13, w and w' exist (assuming $w \cap w' = \Lambda$) such that $\text{N}c`w = \dim_\phi`v$ and $\text{N}c`w + \text{N}c`w' = \dim_\phi`u$. Hence $\dim_\phi`v < \dim_\phi`u$, unless $w'$ is the null class, or unless the numbers are not finite, in which cases $\dim_\phi`v = \dim_\phi`u$ is possible.

*12·23. *Proposition.*—Assuming $(\lambda - \pi) \operatorname{Hp} \phi$, if u and v are subclasses of O_ϕ, and v is not the null class, and $\text{cm}_\phi`v$ is contained in $\text{cm}_\phi`u$, then, if $\dim_\phi`v = \dim_\phi`u$, we have $\text{cm}_\phi`v = \text{cm}_\phi`u$, and conversely. In symbols,

$$\vdash :: (\lambda - \pi) \operatorname{Hp} \phi . \supset .\because v, u \in \text{cls}`\text{O}_\phi . \exists ! v . \text{cm}_\phi`v \subset \text{cm}_\phi`u . \supset :$$
$$\text{cm}_\phi`v = \text{cm}_\phi`u . \equiv . \dim_\phi`v = \dim_\phi`u$$

Proof.—Assuming $\dim_\phi`v = \dim_\phi`u$, and also assuming the notation of the proof of *12·21, then w and w' are such that (1) w is ϕ-equivalent to v and $w \cup w'$ to u, (2) $\text{N}c`w = \dim_\phi`v$ and $\text{N}c`w + \text{N}c`w' = \dim_\phi`u$. Hence, by hypothesis and (2), $\text{N}c`w + \text{N}c`w' = \text{N}c`w$. Also (*cf.* *10·3 and *12·21) $\text{N}c`w + \text{N}c`w' \leqq 3$. Hence $\text{N}c`w' = 0$, that is, $w' = \Lambda$. Hence from (1), $\text{cm}_\phi`u = \text{cm}_\phi`v$. The converse is obvious.

*12·33. *Proposition.*—Assuming $(\lambda - \pi) \operatorname{Hp} \phi$, if u and v are subclasses of O_ϕ, and v is not the null class, and $\text{cm}_\phi`v$ is contained in, but is not identical with, $\text{cm}_\phi`u$, then $\dim_\phi`v$ is less than $\dim_\phi`u$. In symbols,

$$\vdash .\because (\lambda - \pi) \operatorname{Hp} \phi . \supset : u, v \in \text{cls}`\text{O}_\phi . \exists ! v . \text{cm}_\phi`v \subset \text{cm}_\phi`u .$$
$$\text{cm}_\phi`v \neq \text{cm}_\phi`u . \supset . \dim_\phi`v < \dim_\phi`u$$

Proof.—*Cf.* *12·21·23.

The following proposition should be compared with *12·12 :—

*12·37. *Proposition.*—Assuming $(\lambda-\pi)$ Hp ϕ, if u is a subclass of O_ϕ, and is not the null class, there exists a subclass of u which is ϕ-prime and ϕ-equivalent to u. In symbols,

$$\vdash \therefore (\lambda-\pi)\,\mathrm{Hp}\,\phi\,.\supset : u\,\epsilon\,\mathrm{cls}`O_\phi\,.\,\exists!u\,.\supset .\,\exists!(\mathrm{prm}_\phi \cap \mathrm{cls}`u \cap \mathrm{equiv}_\phi`u)$$

Proof.—From *12·12·21, u is either of one, or of two, or of three ϕ-dimensions. If u is of one ϕ-dimension, the conclusion follows from *5·233 and *10·2 and *11·12. If u is of two ϕ-dimensions, then (*cf.* *5·235 and *10·2) any two members of it form a ϕ-prime class, and (*cf.* *12·21) the ϕ-dimension number of this class is not greater than two, and hence (*cf.* *6·25) it is two, and hence (*cf.* *12·23) this subclass is ϕ-equivalent to u. If u is of three ϕ-dimensions, it must contain at least one subclass v consisting of two members, and, as before, v must be ϕ-prime. If v is of three ϕ-dimensions, then (*cf.* *12·23) u and v are ϕ-equivalent. If v is of two ϕ-dimensions, then there is a member of u, x say, which is not a member of $\mathrm{cm}_\phi`v$. Then, either $v \cup \iota`x$ is ϕ-prime and (*cf.* *12·23) ϕ-equivalent to u, or the class composed of x and some one (not necessarily any one) of the members of v is ϕ-prime and ϕ-equivalent to u.

The following proposition should be compared to *3·13 and *12·33 :—

*12·41. *Proposition.*—Assuming $(\lambda-\pi)$ Hp ϕ, if u is ϕ-axial, and v is a subclass of u, and both v and $(u-v)$ are not the null class, then $\dim_\phi`v$ is less than $\dim_\phi`u$. In symbols,

$$\vdash \therefore (\lambda-\pi)\,\mathrm{Hp}\,\phi\,.\supset : u\,\epsilon\,\mathrm{ax}_\phi\,.\,v\,\epsilon\,\mathrm{cls}`u\,.\,\exists!v\,.\,\exists!(u-v)\,.\supset .\,\dim_\phi`v < \dim_\phi`u$$

Proof.—*Cf.* *3·13 and *4·21 and *11·21 and *12·21·23.

The following proposition should be compared to *5·23 :—

*12·42. *Proposition.*—Assuming $(\lambda-\pi)$ Hp ϕ, then, if u is a subclass of O_ϕ and is ϕ-axial, any subclass of u, not the null class, is ϕ-axial. In symbols,

$$\vdash \therefore (\lambda-\pi)\,\mathrm{Hp}\,\phi\,.\supset : u\,\epsilon\,\mathrm{ax}_\phi \cap \mathrm{cls}`O_\phi\,.\,v\,\epsilon\,\mathrm{cls}`u\,.\,\exists!v\,.\supset .\,v\,\epsilon\,\mathrm{ax}_\phi$$

Proof.—From *6·26 we have $\mathrm{N}c`u = \dim_\phi`u$; from *5·23 and *6·25 we have $\mathrm{N}c`v \leqq \dim_\phi`v$. Hence (*cf.* *12·41), if v is not identical with u, we have

$$\mathrm{N}c`v \leqq \dim_\phi`v < \mathrm{N}c`u \quad \ldots\ldots\ldots \quad (1)$$

Firstly, assume that v omits one member of u only. Then $\mathrm{N}c`v+1 = \mathrm{N}c`u$. Hence, from (1), $\mathrm{N}c`v = \dim_\phi`v$, and hence (*cf.* *3·21) v is ϕ-axial.

Secondly, if v omits two members of u, then it is a unit class, and (*cf.* *12·11) is ϕ-axial.

It is convenient to conclude this section (*12) with three theorems which are fundamental to the theory of ϕ-points and of ϕ-planes.

*12·51. *Proposition.*—Assuming $(\lambda-\pi)$ Hp ϕ, if u is of two ϕ-dimensions, and x and y are members of $\mathrm{cm}_\phi`u$, then the class composed of x and y is ϕ-axial and is a subclass of O_ϕ. In symbols,

$$\vdash \therefore (\lambda-\pi)\,\mathrm{Hp}\,\phi\,.\supset : \dim_\phi`u = 2\,.\,x, y\,\epsilon\,\mathrm{cm}_\phi`u\,.\supset .\,\iota`x \cup \iota`y\,\epsilon\,\mathrm{ax}_\phi \cap \mathrm{cls}`O_\phi$$

Proof.—$\mathrm{cm}_\phi{}^\backprime u$ is contained in O_ϕ (*cf.* *4·21·27 and *10·1 and *11·21). If x and y are identical, *cf.* *12·11. If x is distinct from y, then *cf.* *3·21 and *12·11·21·23.

*12·52. *Proposition.*—Assuming $(\lambda - \pi)\,\mathrm{H}p\,\phi$, if u has three members only, and is a self-ϕ-concurrence, and its ϕ-dimension number is three, then u is ϕ-axial and a subclass of O_ϕ. In symbols,

$$\vdash .\therefore (\lambda - \pi)\,\mathrm{H}p\,\phi\,.\supset : u \in 3 \cap \mathrm{conc}_\phi\,.\,\mathrm{dim}_\phi = 3\,.\supset .\, u \in \mathrm{ax}_\phi \cap \mathrm{cls}{}^\backprime \mathrm{O}_\phi$$

Proof.—If v is contained in u and possesses two members, then (*cf.* *3·31·32) v is ϕ-axial, and (*cf.* *12·23) is not ϕ-equivalent to u. Hence (*cf.* *5·231) u is ϕ-prime, and hence (*cf.* *3·22) is ϕ-axial, and also (*cf.* *11·21) is a subclass of O_ϕ.

*12·53. *Proposition.*—Assuming $(\lambda - \pi)\,\mathrm{H}p\,\phi$, if x, y, and z are three distinct entities forming a ϕ-axial class, then the common subclass of $\mathrm{cm}_\phi{}^\backprime(\iota^\backprime x \cup \iota^\backprime y)$ and $\mathrm{cm}_\phi{}^\backprime(\iota^\backprime x \cup \iota^\backprime z)$ is the unit class $\iota^\backprime x$. In symbols,

$$\vdash .\therefore (\lambda - \pi)\,\mathrm{H}p\,\phi\,.\supset : \iota^\backprime x \cup \iota^\backprime y \cup \iota^\backprime z \in 3 \cap \mathrm{ax}_\phi\,.\supset .\,\mathrm{cm}_\phi{}^\backprime(\iota^\backprime x \cup \iota^\backprime y) \cap \mathrm{cm}_\phi{}^\backprime(\iota^\backprime x \cup \iota^\backprime z) = \iota^\backprime x$$

Proof.—*Cf.* *4·21·27 and *10·2 and *12·23·42.

*13. *On ϕ-Maximal Classes and Self-ϕ-Concurrences.*

*13·11. *Proposition.*—Assuming $(\lambda - \pi)\,\mathrm{H}p\,\phi$, if p is a ϕ-maximal class and a subclass of O_ϕ, and q is a subclass of p, not the null class, then q is a ϕ-maximal class. In symbols,

$$\vdash .\therefore (\lambda - \pi)\,\mathrm{H}p\,\phi\,.\supset : p \in \mathrm{mx}_\phi \cap \mathrm{cls}{}^\backprime \mathrm{O}_\phi\,.\,q \in \mathrm{cls}{}^\backprime p\,.\,\exists ! p\,.\supset .\,q \in \mathrm{mx}_\phi$$

Proof.—The class q must (*cf.* *10·3 and *12·21) be of one, or two, or three ϕ-dimensions. If the ϕ-dimension number of q is one or two, then (*cf.* *10·2 and *12·11·51) q is a ϕ-maximal class. If the ϕ-dimension number of q is three, then (*cf.* *12·23) q is ϕ-equivalent to p. Hence if v be a subclass of q, which is ϕ-prime and ϕ-equivalent to q, it is ϕ-equivalent to p, and hence (*cf.* *3·23) it is ϕ-axial. Hence (*cf.* *3·23 and *12·37) q is ϕ-maximal.

*13·31. *Proposition.*—Assuming $(\lambda - \pi)\,\mathrm{H}p\,\phi$, if u is a self-ϕ-concurrence and a subclass of O_ϕ, then u is a ϕ-maximal class. In symbols,

$$\vdash .\therefore (\lambda - \pi)\,\mathrm{H}p\,\phi\,.\supset : u \in \mathrm{conc}_\phi \cap \mathrm{cls}{}^\backprime \mathrm{O}_\phi\,.\supset .\,u \in \mathrm{mx}_\phi$$

Proof.—There exists (*cf.* *12·37) a subclass (v) of u, which is ϕ-prime and ϕ-equivalent to u. If v is a unit class, then (*cf.* *10·2 and *12·11) v and u are identical, and u is of one ϕ-dimension and ϕ-maximal. If v is a couple, then (*cf.* *3·31·32) v is ϕ-axial, and (*cf.* *3·22·23) u is of two ϕ-dimensions and is ϕ-maximal. If v is composed of three members, then u is of three ϕ-dimensions, and neither of the previous cases can hold. Hence again u is ϕ-maximal.

*13·32. *Proposition.*—Assuming $(\lambda - \pi)$ H$p\,\phi$, all subclasses of O_ϕ which are ϕ-maximal are self-ϕ-concurrent, and conversely. In symbols,

$$\vdash : (\lambda - \pi)\,\mathrm{H}p\,\phi \,.\supset.\, \mathrm{mx}_\phi \cap \mathrm{cls}'\,O_\phi = \mathrm{conc}_\phi \cap \mathrm{cls}'\,O_\phi$$

Proof.—*Cf.* *13·11·31.

*14. *On Points.*

*14·11. Assuming $(\lambda - \rho)$ H$p\,\phi$, every ϕ-point is a self-ϕ-concurrence and a subclass of O_ϕ. In symbols,

$$\vdash : (\lambda - \rho)\,\mathrm{H}p\,\phi \,.\supset.\, \mathrm{pnt}_\phi \subset \mathrm{conc}_\phi \cap \mathrm{cls}'\,O_\phi$$

Proof.—Every ϕ-point (*cf.* *11·21) is a subclass of O_ϕ. Again let x and y be two distinct members of a ϕ-point P. Then (*cf.* *3·42) a, b, c exist such that $\iota' a \cup \iota' b \cup \iota' c$ is ϕ-axial and of three dimensions, and x and y are each ϕ-concurrent with each of a, b, and c. Hence (*cf.* *12·53) at least one pair of a, b, and c exist (say a and b) such that $\iota' x \cup \iota' a \cup \iota' b$ and $\iota' y \cup \iota' a \cup \iota' b$ are both three ϕ-dimensional and self-ϕ-concurrences. Hence (*cf.* *12·52) $\iota' x \cup \iota' a \cup \iota' b$ and $\iota' x \cup \iota' a \cup \iota' b$ are both ϕ-axial. Hence (*cf.* *10·5 and *13·32) $\iota' x \cup \iota' y$ is ϕ-axial. Hence P is a self-ϕ-concurrence.

*14·12. *Proposition.*—Assuming $(\lambda - \rho)$ H$p\,\phi$, every ϕ-point is ϕ-maximal. In symbols,

$$\vdash : (\lambda - \rho)\,\mathrm{H}p\,\phi \,.\supset.\, \mathrm{pnt}_\phi \subset \mathrm{mx}_\phi$$

Proof.—*Cf.* *13·32 and *14·11.

*14·13. *Proposition.*—Assuming $(\lambda - \rho)$ H$p\,\phi$, if P is a ϕ-point, then P is the ϕ-concurrence of O_ϕ with P. In symbols,

$$\vdash .\because (\lambda - \rho)\,\mathrm{H}p\,\phi \,.\supset:\, \mathrm{P} \in \mathrm{pnt}_\phi \,.\supset.\, \mathrm{P} = \tilde{O}_\phi{}'\,\mathrm{P}$$

Proof.—*Cf.* *3·42 and *8·21·22 and *14·11.

*14·14. *Proposition.*—Assuming $(\lambda - \pi)$ H$p\,\phi$, if P is a ϕ-point, it possesses at least three members. In symbols,

$$\vdash .\because (\lambda - \pi)\,\mathrm{H}p\,\phi \,.\supset:\, \mathrm{P} \in \mathrm{pnt}_\phi \,.\supset.\, Nc'\,\mathrm{P} \geqq 3$$

Proof.—P possesses (*cf.* *3·42) every member of O_ϕ which is ϕ-concurrent with each of a certain ϕ-axial set of three members. Hence (*cf.* *12·42) P possesses this set of three members.

*14·21. *Proposition.*—Assuming $(\lambda - \rho)$ H$p\,\phi$, ϕ-points with more than one member in common are identical. In symbols,

$$\vdash .\because (\lambda - \rho)\,\mathrm{H}p\,\phi \,.\supset:\, \mathrm{P}, \mathrm{Q} \in \mathrm{pnt}_\phi \,.\, Nc'(\mathrm{P} \cap \mathrm{Q}) > 1 \,.\supset.\, \mathrm{P} = \mathrm{Q}$$

Proof.—Let a and b be two distinct members of $\mathrm{P} \cap \mathrm{Q}$. Then (*cf.* *3·42 and *12·52 and *14·11) c and d exist, such that c is a member of P and d of Q, and

$\iota`a \cup \iota`b \cup \iota`c$ and $\iota`a \cup \iota`b \cup \iota`d$ are both of them three ϕ-dimensional and ϕ-axial. Hence (*cf.* *10·5 and *13·32) d is a member of $\tilde{O}_\phi`(\iota`a \cup \iota`b \cup \iota`c)$, and hence (*cf.* *3·42 and *14·11·13) d is a member of P. Thus P and Q are identical.

*16. *On ϕ-Planes.*

*16·11. *Proposition.*—Assuming $(\lambda-\pi)\,\mathrm{H}p\,\phi$, every ϕ-plane is ϕ-maximal, self-ϕ-concurrent, and a subclass of O_ϕ. In symbols,

$$\vdash : (\lambda-\pi)\,\mathrm{H}p\,\phi\,.\supset.\,\mathrm{ple}_\phi \subset \mathrm{mx}_\phi \cap \mathrm{conc}_\phi \cap \mathrm{cls}`O_\phi$$

Proof.—*Cf.* *12·21·23·51.

*16·21. *Proposition.*—Assuming $(\lambda-\pi)\,\mathrm{H}p\,\phi$, ϕ-planes with more than one member in common are identical. In symbols,

$$\vdash .\,\therefore (\lambda-\pi)\,\mathrm{H}p\,\phi\,.\supset: p, q \,\epsilon\, \mathrm{ple}_\phi\,.\,\mathrm{N}c`(p \cap q) > 1\,.\supset.\, p = q$$

Proof.—Cf. *12·23 and *16·11.

*16·31. *Proposition.*—Assuming $(\lambda-\pi)\,\mathrm{H}p\,\phi$, every self-$\phi$-concurrence is either ϕ-copunctual or ϕ-coplanar. In symbols,

$$\vdash .\,\therefore (\lambda-\pi)\,\mathrm{H}p\,\phi\,.\supset: u \,\epsilon\, \mathrm{conc}_\phi\,.\supset.\,\mathrm{copnt}_\phi!u\,.\vee.\,\mathrm{cople}_\phi!u$$

Proof.—A proof is only required when u is of three ϕ-dimensions. Then a, b, c exist, such that they are three distinct members of u and are not a ϕ-coplanar class. Hence (*cf.* *12·52) they form a ϕ-axial class of three members. Hence (*cf.* *3·42) u is, in this case, ϕ-copunctual.

*16·32. *Proposition.*—Assuming $(\lambda-\rho)\,\mathrm{H}p\,\phi$, if p is a ϕ-plane, and P and Q are distinct ϕ-points, and p and P have common members, and also p and Q, then the member (if any) common to P and Q is a member of p. In symbols,

$$\vdash .\,\therefore (\lambda-\rho)\,\mathrm{H}p\,\phi\,.\supset: p \,\epsilon\, \mathrm{ple}_\phi\,.\,\mathrm{P}, \mathrm{Q} \,\epsilon\, \mathrm{pnt}_\phi\,.\,\mathrm{P} \neq \mathrm{Q}\,.\,\exists!(p \cap \mathrm{P})\,.\,\exists!(p \cap \mathrm{Q})\,.\supset.\,\mathrm{P} \cap \mathrm{Q} \subset p$$

Proof.—If $\mathrm{P} \cap \mathrm{Q}$ is the null class, then $\mathrm{P} \cap \mathrm{Q}$ is contained in p. If $\mathrm{P} \cap \mathrm{Q}$ is not null, let c be a member; also let a and b be, respectively, members of $p \cap \mathrm{P}$ and of $p \cap \mathrm{Q}$, which exist by hypothesis. (i) If c is identical with a or b, then (*cf.* *14·21) $\mathrm{P} \cap \mathrm{Q}$ is contained in p. Again (ii) if c is not identical with a or b, then (*cf.* *14·11 and *16·11) a, b and c form a self-ϕ-concurrence. Hence (*cf.* *16·31) this class is either ϕ-copunctual or ϕ-coplanar. If the class is ϕ-copunctual, then (*cf.* *14·21) P and Q are identical. Hence it is ϕ-coplanar, and hence (*cf.* *16·21) c is a member of p.

*16·33. *Proposition.*—Assuming $(\lambda-\rho)\mathrm{H}p\,\phi$, if P is a ϕ-point and p and q are distinct ϕ-planes, and P and p have common members, and so, also, have P and q, then the member (if any) common to p and q is a member of P. In symbols,

$$\vdash .\,\therefore (\lambda-\rho)\,\mathrm{H}p\,\phi\,.\supset: \mathrm{P} \,\epsilon\, \mathrm{pnt}_\phi\,.\,p, q \,\epsilon\, \mathrm{ple}_\phi\,.\,p \neq q\,.\,\exists!(\mathrm{P} \cap p)\,.\,\exists!(\mathrm{P} \cap q)\,.\supset.\,p \cap q \subset \mathrm{P}$$

Proof.—The proof is in all respects similar to that of *16·32.

*16·42. *Proposition.*—Assuming $(\lambda - \rho)\, \mathrm{H}p\, \phi$, if p is a ϕ-plane and is not ϕ-copunctual, then p is the ϕ-concurrence of O_ϕ with p. In symbols,

$$\vdash .\because (\lambda - \rho)\, \mathrm{H}p\, \phi . \supset : p \in \mathrm{ple}_\phi . \sim \mathrm{copnt}_\phi ! \, p . \supset . p = \tilde{\mathrm{O}}_\phi ` p$$

Proof.—From *8·21 and *16·11 we have

$$p \subset \tilde{\mathrm{O}}_\phi ` p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let x be any member of the ϕ-concurrence of O_ϕ with p. Hence (*cf.* *16·11) $p \cup \iota`x$ is a self-$\phi$-concurrence. Hence (*cf.* 16·31) $p \cup \iota`x$ is either ϕ-copunctual or ϕ-coplanar. But on the first alternative p is ϕ-copunctual. Hence $p \cup \iota`x$ is ϕ-coplanar. Hence x is a member of p. Hence from (1) the proposition follows.

Note.—In Concept V. the hypothesis of *16·42, that a ϕ-plane is not copunctual, is verified (*cf.* *28·11), where ϕ represents "homaloty," and the axioms of that concept are assumed.

Summary of the Complete Development of this Subject.—By the use of further axioms the whole theory of projective geometry, apart from "order" and apart from FANO's axiom† respecting the distinction of harmonic conjugates, can be proved for ϕ-points and the associated geometrical entities. Then FANO's axiom can be added, and the theory of order and continuity can be introduced, as in PIERI's memoir (*loc. cit.*). In the sequel a somewhat different line of development is adopted, suitable for the special ideas of Concept V.

(ii) CONCEPT V.

This concept is linear and monistic. It makes use both of the theory of interpoints and of the theory of dimensions. The points are classes of objective reals, and disintegrate from instant to instant. The corpuscles are capable of various and complicated structures, and are thus well fitted to bear the weight of modern physical ideas. The concept is Leibnizian, and only requires one extraneous relation for the same purposes as that of Concept III.

The essential relation is the pentadic relation $\mathrm{R}`(abcdt)$, as explained at the commencement of Part III. The four first terms, namely, a, b, c, d, are objective reals and are mutually distinct, the fifth term is an instant of time.

The relation $\mathrm{R}`(abcdt)$ can be read, *a intersects b, c, d in the order bcd at the instant t.* In this concept *copunctual* objective reals do *not* necessarily *intersect*, though two intersecting objective reals are necessarily copunctual. The relation of *intersection* is not to be limited in properties by the mere geometrical suggestion of its technical name.

Since points are defined by the aid of the theory of dimensions, it follows (*cf.* note

† *Cf.* PIERI, *loc. cit.*

to *3·42) that the geometry cannot be of less than three dimensions. Hence in this concept geometry of three dimensions occupies a position of unique simplicity.

The points at infinity, here called *cogredient points*, are points in exactly the same sense as the other points. They are defined by a property not hitherto taken as fundamental. The properties of cogredient points play an essential part in the construction of a relation which assigns an *order* to the points on any straight line.

*20. *Definitions.*

*20·11. *Definition.*—An objective real p is *doubly secant* with a class u at an instant t if there exist two objective reals, members of u (x and y, say), which are both intersected by p at the instant t, and are such that there exists no interpoint on p of which x and y are both members. The symbol $(\widetilde{uu})_{\mathrm{R}t}!p$ will denote that p is doubly secant with u at the instant t. The symbolic definition is

$$(\widetilde{uu})_{\mathrm{R}t}!p \,.=:(\exists x, y) \,.\, x \neq y \,.\, x, y \in u \cap \mathrm{R}^{;}(p;;;t) \,.\sim (\exists v) \,.\, v \in \mathrm{R}^{;}(p???t) \,.\, x, y \in v \quad \mathrm{Df}$$

*20·12. *Definition.*—A class u is *homalous* at an instant t, *either* when a necessary and sufficient condition, that x should be a member of u, is that x should be doubly secant with u, *or* when u is a unit class contained in $\mathrm{R}^{;}(;;;;t)$. The symbol $\mu_{\mathrm{R}t}!u$ will denote that u has the property of homaloty at the instant t. The symbolic definition is

$$\mu_{\mathrm{R}t}!u \,.=\therefore x \in u \,.\equiv_x .\, (\widetilde{uu})_{\mathrm{R}t}!x : \vee : u \in 1 \cap \mathrm{cls}`\mathrm{R}^{;}(;;;;t) \qquad \mathrm{Df}$$

This property ($\mu_{\mathrm{R}t}$) of homaloty will now be taken as the special value of ϕ, to which the theory of dimensions will be applied. The common $\mu_{\mathrm{R}t}$-subregion for u is denoted, according to the definition of *3·12, by $\mathrm{cm}_{\mu_{\mathrm{R}t}}`u$. But a suffix to a suffix will be avoided by using the simpler symbol $\mathrm{cm}_{\mathrm{R}t}`u$, and similarly for the other entities defined in *3. Thus the following symbols are also defined, namely,

$$\mathrm{O}_{\mathrm{R}t}, \quad \mathrm{equiv}_{\mathrm{R}t}`u, \quad \mathrm{prm}_{\mathrm{R}t}, \quad \mathrm{dim}_{\mathrm{R}t}, \quad \mathrm{ax}_{\mathrm{R}t}, \quad \mathrm{mx}_{\mathrm{R}t}, \quad \tilde{u}_{\mathrm{R}t}`v, \quad \mathrm{conc}_{\mathrm{R}t}, \quad \mathrm{ple}_{\mathrm{R}t},$$
$$\mathrm{pnt}_{\mathrm{R}t}, \quad \mathrm{cople}_{\mathrm{R}t}!u, \quad \mathrm{copnt}_{\mathrm{R}t}!u.$$

With regard to the nomenclature, the term "ϕ-equivalence" should be particularized into "homaloty-equivalence," and "ϕ-prime" into "homaloty-prime," and so on. But, except where confusion is likely to occur, the term "homaloty" will be dropped; and the terms "equivalence," "prime," "dimensions," "axial," "maximal," "concurrence of u with v," "self-concurrence," "plane," "point," "coplanar," "copunctual" will be used in the senses defined in *3, with ϕ particularized into homality.

Elucidatory Note.—This definition of *homaloty* should be compared with the definition of the *flatness* of a class of punctual lines which has been used in the elucidatory notes of *3. Thus a class of punctual lines is flat, *either* when it is a unit class whose single member is a straight line, *or* when it is a necessary and

sufficient condition of a straight line x being a member of it, that x should meet two members of the class in points which are not their point of meeting (if they have a point of meeting). Owing to the fact that "intersection" (as used here) is wider in intension and narrower in extension than the idea of the "meeting" of two punctual lines, two punctual lines may "meet" without the corresponding objective reals "intersecting." The result is that homaloty and flatness have some different properties, for example, *cf.* *21·21.

*20·21. *Definition.*—The *punctual associate* of a class u is the class of those points which have a member in common with u. The punctual associate of u is denoted by $\text{ass}_{\text{R}t}\text{‘}u$. The definition in symbols is

$$\text{ass}_{\text{R}t}\text{‘}u = \dot{\text{P}}\{\text{P} \in \text{pnt}_{\text{R}t} \,.\, \exists\,!(\text{P} \cap u)\} \qquad \text{Df}$$

Note.—The punctual associate of the class $\iota\text{‘}a$, *where a is an objective real*, will be called the punctual associate of a. Its symbol is $\text{ass}_{\text{R}t}\text{‘}\iota\text{‘}a$.

*20·22. *Definition.*—A *punctual line* is a class of points such that there exist two planes, p and q, which are distinct and are such that the class of points is the common subclass of the punctual associates of p and q. The class of punctual lines at any instant t is denoted by $\text{lin}_{\text{R}t}$. The symbolic definition is

$$\text{lin}_{\text{R}t} = \dot{m}\{(\exists p, q) \,.\, p, q \in \text{ple}_{\text{R}t} \,.\, p \neq q \,.\, m = \text{ass}_{\text{R}t}\text{‘}p \cap \text{ass}_{\text{R}t}\text{‘}q\} \qquad \text{Df}$$

Note.—Those punctual lines which are not "lines at infinity" (to be explained later) will be proved as the result of the axioms to be the punctual associates of the various objective reals.

*20·23. *Definition.*—The point, if there is one and one only, which contains a class u is called the *dominant point of u.* The dominant point of u is denoted by $u_{\text{R}t}$. The symbolic definition is

$$u_{\text{R}t} = (\imath p)\{p \in \text{pnt}_{\text{R}t} \,.\, u \in \text{cls}\text{‘}p\} \qquad \text{Df}$$

Note.—The idea of a dominant point obtains its importance from the fact that, according to the axioms given below, each interpoint is contained in one and only one point.

*20·231. *Definition.*—The *nonsecant part of u* is that subclass of u of which no member is a member of any interpoint which is a subclass of u. The nonsecant part of u is denoted by $\text{nsc}_{\text{R}t}\text{‘}u$. The symbolic definition is

$$\text{nsc}_{\text{R}t}\text{‘}u = \dot{x}\{x \in u \,.\, \sim (\exists v) \,.\, v \in \text{intpnt}_{\text{R}t} \cap \text{cls}\text{‘}u \,.\, x \in v\} \qquad \text{Df}$$

Note.—This definition takes its importance from the fact that (assuming the subsequent axioms) a point in general consists of a nonsecant part and of a part made up of interpoints contained in it. Either the interpoints or the nonsecant part may be wholly absent.

*20·232. *Definition.*—A class of points is called a *Figure.*

*20·233. *Definition.*—A point, which is a member of a figure, will be said to *lie in* that figure.

*20·234. *Definition.*—A point, which lies in the punctual associate of a class of objective reals, will be said to *be on*, or *upon*, that class.

*20·235. *Definition.*—A punctual line is said to *join* two points if both the points lie in it.

*20·236. *Definition.*—Two punctual lines, which possess a common point, will be said to *meet at* that point. Similarly, any two classes of points will be said to *meet in* their common subclass, and this subclass will be called their *meeting.*

*20·24. *Definition.*—A class of points is called *collinear* if there exist a punctual line in which they all lie. The symbol $\mathrm{coll}_{\mathrm{R}t}!u$ will denote that u is a class of collinear points at the instant t. The symbolic definition is

$$\mathrm{coll}_{\mathrm{R}t}!u \,.\!=.\, (\exists m) \,.\, m \in \mathrm{lin}_{\mathrm{R}t} \,.\, u \in \mathrm{cls}`m \qquad \mathrm{Df}$$

*20·31. *Definition.*—Two figures are *in perspective* if (i) they have a one-one correspondence to each other, (ii) the joint figure formed by the two figures combined is not collinear, and (iii) there exists a point (the centre of perspective) which lies in every punctual line joining two *distinct* corresponding points. The statement that u and v are in perspective with each other at the instant t, and that S is the requisite one-one correspondence, will be denoted by $u\,(\mathrm{S\ persp})_{\mathrm{R}t}\, v$. The symbolic definition is

$$u(\mathrm{S\ persp})_{\mathrm{R}t}\, v \,.\!=:\, u, v \in \mathrm{cls}`\mathrm{pnt}_{\mathrm{R}t} \,.\, \sim \mathrm{coll}_{\mathrm{R}t}!(u \cup v) \,.\, \mathrm{S} \in 1 \rightarrow 1 \,.\, u = \mathrm{S}`(;\cdot) \,.\, v = \mathrm{S}`(\cdot;) :$$
$$(\exists \mathrm{V}) : m \in \mathrm{lin}_{\mathrm{R}t} \,.\, \mathrm{S}`(\mathrm{AA}') \,.\, \mathrm{A} \neq \mathrm{A}' \,.\, \mathrm{A}, \mathrm{A}' \in m \,.\, \supset_{m,\mathrm{A},\mathrm{A}'} .\, \mathrm{V} \in m \qquad \mathrm{Df}$$

*20·32. *Definition.*—The symbol $[\mathrm{AB}]\,\mathrm{persp}_{\mathrm{R}t}\,[\mathrm{A}'\mathrm{B}']$ denotes that A, B, A′, B′ are points, and that the figure formed by A and B is in perspective with the figure formed by A′ and B′, and that the one-one correspondence of the perspective is of A to A′ and of B to B′. Also $[\mathrm{ABC}]\,\mathrm{persp}_{\mathrm{R}t}\,[\mathrm{A}'\mathrm{B}'\mathrm{C}']$ has a similar meaning, and so on. In symbols,

$$[\mathrm{AB}]\,\mathrm{persp}_{\mathrm{R}t}\,[\mathrm{A}'\mathrm{B}'] \,.\!=.\, (\exists \mathrm{S}) \,.\, (\iota`\mathrm{A} \cup \iota`\mathrm{B})\,(\mathrm{S\ persp})_{\mathrm{R}t}\,(\iota`\mathrm{A}' \cup \iota`\mathrm{B}') \,.\, \mathrm{S}`(\mathrm{AA}') \,.\, \mathrm{S}`(\mathrm{BB}') \qquad \mathrm{Df}$$

$$[\mathrm{ABC}]\,\mathrm{persp}_{\mathrm{R}t}\,[\mathrm{A}'\mathrm{B}'\mathrm{C}'] \,.\!=.\, (\exists \mathrm{S}) \,.\, (\iota`\mathrm{A} \cup \iota`\mathrm{B} \cup \iota`\mathrm{C})\,(\mathrm{S\ persp})_{\mathrm{R}t}\,(\iota`\mathrm{A}' \cup \iota`\mathrm{B}' \cup \iota`\mathrm{C}') \,.$$
$$\mathrm{S}`(\mathrm{AA}') \,.\, \mathrm{S}`(\mathrm{BB}') \,.\, \mathrm{S}`(\mathrm{CC}') \qquad \mathrm{Df}$$

*20·33. *Definition.*—The symbol $u\,\mathrm{persp}_{\mathrm{R}t}\,v$ denotes that there exists a one-one relation S such that, at the instant t, u is in perspective with v and S is the requisite one-one correspondence. In symbols,

$$u\,\mathrm{persp}_{\mathrm{R}t}\,v \,.\!=.\, (\exists \mathrm{S}) \,.\, u\,(\mathrm{S\ persp})_{\mathrm{R}t}\,v \qquad \mathrm{Df}$$

*20·41. *Definition.*—Two objective reals, a and c, are called *cogredient* at an instant t when (1) if u, v, w are three interpoints on a, and u', v', w' are three interpoints on c, and the dominant points $u_{\mathrm{R}t}$, $v_{\mathrm{R}t}$, $w_{\mathrm{R}t}$ are a trio of points in

perspective with the trio of dominant points u'_{Rt}, v'_{Rt}, w'_{Rt}, then the interpoint relation (R_{in}), if it arranges either trio of interpoints in an interpoint order, arranges both trios of interpoints in the same interpoint order (*i.e.*, either uvw and $u'v'w'$, or vwu and $v'w'u'$, or so on), and (2) there exist three interpoints u, v, w on a in the interpoint order uvw, and three interpoints u', v', w' on c such that u_{Rt}, v_{Rt}, w_{Rt} are in perspective with u'_{Rt}, v'_{Rt}, w'_{Rt}. The symbol $\text{cogrd}_{Rt}`a$ denotes the class of objective reals cogredient with α. The symbolic definition is

$$\begin{aligned} \text{cogrd}_{Rt}`a = \hat{x}\{(u, v, w, u', v', w') : u, v, w \in R^{;}(a???t) \,.\, u', v', w' \in R^{;}(x???t) \,. \\ [u_{Rt}v_{Rt}w_{Rt}]\,\text{persp}_{Rt}\,[u'_{Rt}v'_{Rt}w'_{Rt}] \,.\supset.\, R_{in}{}^{;}(uvwt) \equiv R_{in}{}^{;}(u'v'w't) : \\ (\exists\, u, v, w, u', v', w') \,.\, u, v, w \in R^{;}(a???t) \,.\, u', v', w' \in R^{;}(x???t) \,. \\ R_{in}{}^{;}(uvwt) \,.\, [u_{Rt}v_{Rt}w_{Rt}]\,\text{persp}_{Rt}\,[u'_{Rt}v'_{Rt}w'_{Rt}]\} \qquad \text{Df} \end{aligned}$$

Note.—The class $\text{cogrd}_{Rt}`a$ does not include a itself (*cf.* *27·43). It will be noticed that universal preservation of order by ranges in perspective on a pair of lines is a characteristic of a pair of parallel lines in Euclidean space, and of nonsecant lines in hyperbolic space. The choice of this property for the definition of parallelism (or nonsecancy) arises from the facts that (1) any two coplanar objective reals are copunctual (according to the subsequent axioms), so that the property of nonsecancy (in its ordinary acceptation) is not available, (2) we do not wish to make "cogredience" synonymous with "nonintersection" (using "intersection" in the special sense here defined), as this would impose an unnecessary limitation on the concept. The idea of cogredience is an essential element in the definition of a relation which, with the aid of axioms, distributes the points in any punctual line into an order.

*20·42. *Definition.*—A *Cogredient Point* is the class of objective reals cogredient with some objective real a, together with a itself. The symbol ∞_{Rt} denotes the class of cogredient points at the instant t. The definition in symbols is

$$\infty_{Rt} = \hat{u}\{(\exists a) \,.\, a \in O_{Rt} \,.\, u = \iota`a \cup \text{cogrd}_{Rt}`a\} \qquad \text{Df}$$

Note.—In the case of Euclidean geometry, which is the only case considered here, each cogredient point is a point according to the definition of *3·42. The present definition would be very inconvenient, unless this were the case. The symbol ∞_{Rt} is reminiscent of the fact that the cogredient points are the points at infinity.

*20·51. *Definition.*—The *Point-Ordering Relation* is a tetradic relation holding between three points and an instant of time. Its symbol is R_{pn}, and $R_{pn}{}^{;}(ABCt)$ is defined to mean that, at the instant t, (1) A, B, C are non-cogredient points upon the same objective real, a say, and (2) there exist an objective real x and three interpoints u, v, w on x such that (i) x is cogredient with a, and (ii) u, v, w are in the interpoint order uvw, and (iii) A, B, C are in perspective with the dominant points u_{Rt}, v_{Rt}, w_{Rt}. The definition in symbols is

$$\mathrm{R_{pn}};(\mathrm{ABC}t) \,.\!=.\, \mathrm{A, B, C} \in \mathrm{pnt_{R}}_t - \infty_{\mathrm{R}t} \,.\, (\exists a, x, u, v, w) \,.\, a \in \mathrm{A} \cap \mathrm{B} \cap \mathrm{C} \,.\, x \in \mathrm{cogrd_{R}}_t\text{‘}a \,.$$

$$u, v, w \in \mathrm{R};(x???t) \,.\, \mathrm{R_{in}};(uvwt) \,.\, [\mathrm{ABC}]\, \mathrm{persp}_{\mathrm{R}t}\, [u_{\mathrm{R}t} v_{\mathrm{R}t} w_{\mathrm{R}t}] \quad \mathrm{Df}$$

Note.—Since (*cf.* *27·43) x in the above definition is distinct from a, three collinear (*i.e.*, on a) points, A, B, C, cannot directly take their point-order from three interpoints which they themselves may severally contain (*cf.* however *21·51). The point-order of A, B, and C must arise from the order communicated (in a sense) to a copunctual pencil of three punctual lines by three interpoints contained respectively in points in these lines, and all three interpoints possessing an objective real (x) in common. The punctual lines of this pencil must possess A, B, and C respectively. This intervention of a pencil for the communication of point-order is necessary for the comparison of the orders of different ranges. If the apparently simpler plan is adopted, inextricable difficulties seem to arise. Also it will be remembered that not every point will necessarily contain an interpoint.

*20·61. *Definition.*—A *Punctual Plane* is a figure which is *either* the punctual associate of some plane, *or* is the class $\infty_{\mathrm{R}t}$. The class of punctual planes is denoted by $\mathrm{pple}_{\mathrm{R}t}$. The definition in symbols is

$$\mathrm{pple}_{\mathrm{R}t} = \mathrm{ass}_{\mathrm{R}t}\text{“}\mathrm{ple}_{\mathrm{R}t} \cup \iota\text{‘}\,\infty_{\mathrm{R}t} \qquad \mathrm{Df}$$

Note.—This definition is only convenient for Euclidean geometry.

*20·72. *Definition.*—A figure is called *Punctually Coplanar* if there is a punctual plane containing it. The symbol $\mathrm{copple}_{\mathrm{R}t}!u$ will denote that u is a punctually coplanar figure at the instant t. In symbols,

$$\mathrm{copple}_{\mathrm{R}t}!u \,.\!=.\, (\exists p) \,.\, p \in \mathrm{pple}_{\mathrm{R}t} \,.\, u \in \mathrm{cls}\text{‘}p \qquad \mathrm{Df}$$

Note.—This definition should be compared with that of $\mathrm{cople}_{\mathrm{R}t}!u$ in *3·43.

*21. *General Deductions.*

*21·01. *Proposition.*—All the general deductions in the theory of dimensions, namely, *4 to *8, hold.

The following propositions, dependent on the special definition of homaloty, also hold :—

*21·11. *Proposition.*—$\mathrm{O}_{\mathrm{R}t}$ is the class $\mathrm{R};(;;;;t)$. In symbols,

$$\vdash .\, \mathrm{O}_{\mathrm{R}t} = \mathrm{R};(;;;;t)$$

Proof.—*Cf.* *20·12.

Note.—If t is not an instant of time, the classes $\mathrm{O}_{\mathrm{R}t}$ and $\mathrm{R};(;;;;t)$ are both the null class, and are thus identical. Accordingly the hypothesis, $t \in \mathrm{T}$, is not required in this proposition. A similar explanation of the absence of the hypothesis, $t \in \mathrm{T}$, holds for many other propositions.

*21·21. *Proposition.*—An objective real, which is doubly secant with the common subregion for u, is a member of the common subregion for u. In symbols,

$$\vdash : w = \mathrm{cm}_{\mathrm{R}t}\text{‘}u \,.\, (\widetilde{ww})_{\mathrm{R}t}!x \,.\supset.\, x \in \mathrm{cm}_{\mathrm{R}t}\text{‘}u$$

Proof.—*Cf.* *3·12 and *20·11.

Note.—The converse is not in general true, namely, that, if u is a class of objective reals, and x is a member of $\mathrm{cm}_{\mathrm{R}t}\text{‘}u$, then x is doubly secant with $\mathrm{cm}_{\mathrm{R}t}\text{‘}u$. Nor does this converse follow from subsequent axioms. In the absence of this converse proposition the properties of homaloty differ from those of "flatness" for classes of punctual lines. For if u is a class of punctual lines, and ϕ stands for the property of flatness, then $\mathrm{cm}_{\phi}\text{‘}u$ is flat.

*21·31. *Proposition.*—The proposition $\mu\,\mathrm{H}p\,\phi$ is true when $\mu_{\mathrm{R}t}$ is substituted for ϕ. In symbols,

$$\vdash .\, \mu\,\mathrm{H}p\,\mu_{\mathrm{R}t}$$

Proof.—*Cf.* *10·2 and *20·12.

*21·41. *Proposition.*—If a is an objective real cogredient with c, then c is cogredient with a. In symbols,

$$\vdash : a \in \mathrm{cogrd}_{\mathrm{R}t}\text{‘}a \,.\equiv.\, c \in \mathrm{cogrd}_{\mathrm{R}t}\text{‘}a$$

Proof.—*Cf.* *20·41.

*21·51. *Proposition.*—If u, v, w be three interpoints, possessing the same objective real, and with dominant points $u_{\mathrm{R}t}$, $v_{\mathrm{R}t}$, $w_{\mathrm{R}t}$, then $\mathrm{R}_{\mathrm{pn}}\text{;}(u_{\mathrm{R}t}v_{\mathrm{R}t}w_{\mathrm{R}t}t)$ implies $\mathrm{R}_{\mathrm{in}}\text{;}(uvwt)$. In symbols,

$$\vdash .\!\cdot.\, u, v, w \in \mathrm{R}\text{;}(a???t) \,.\supset:\, \mathrm{R}_{\mathrm{pn}}\text{;}(u_{\mathrm{R}t}v_{\mathrm{R}t}w_{\mathrm{R}t}t) \,.\supset.\, \mathrm{R}_{\mathrm{in}}\text{;}(uvwt)$$

Proof.—By definition (*cf.* *20·51) $\mathrm{R}_{\mathrm{pn}}\text{;}(u_{\mathrm{R}t}v_{\mathrm{R}t}w_{\mathrm{R}t}t)$ implies (1) the existence of an objective real x, cogredient with a, and also of three interpoints, u', v', w', all possessing x, and (2) that $\mathrm{R}_{\mathrm{in}}\text{;}(u'v'w't)$, and (3) that u', v', w' are contained in dominant points $u'_{\mathrm{R}t}$, $v'_{\mathrm{R}t}$, $w'_{\mathrm{R}t}$ in perspective with $u_{\mathrm{R}t}$, $v_{\mathrm{R}t}$, $w_{\mathrm{R}t}$. Hence by the definition of cogredience (*cf.* *20·41) also $\mathrm{R}_{\mathrm{in}}\text{;}(uvwt)$ holds.

*22. *The Axioms.*—Just as in Concept III., the axiom of persistence (*cf.* *22·1) does not enter into the geometrical reasoning, but it is essential to the physical side of the concept.

*22·1. I Hp R is the statement that, *if t be an instant of time, O is contained in* $\mathrm{O}_{\mathrm{R}t}$. In symbols,

$$\mathrm{I\,H}p\,\mathrm{R} \,.=:\, t \in \mathrm{T} \,.\supset_t.\, \mathrm{O} \subset \mathrm{O}_{\mathrm{R}t} \qquad \text{Df}$$

The next four axioms, viz. (II.–V.) Hp R, are the axioms of order. They have already been explained in *1·51·52·53·54.

*22·21. $\mathrm{II\,H}p\,\mathrm{R} = \alpha\,\mathrm{H}p\,\mathrm{R}$ Df

*22·22. $\mathrm{III\,H}p\,\mathrm{R} = \beta\,\mathrm{H}p\,\mathrm{R}$ Df

*22·23. $\mathrm{IV\,H}p\,\mathrm{R} = \gamma\,\mathrm{H}p\,\mathrm{R}$ Df

*22·24. $\mathrm{V\,H}p\,\mathrm{R} = \delta\,\mathrm{H}p\,\mathrm{R}$ Df

The next three axioms, viz. (VI.–VIII.) Hp R, are the axioms establishing the relation of interpoints to points. Intpnt Hp R has been defined in *1·41.

*22·31. VI Hp R = intpnt Hp R Df

*22·32. VII Hp R is the statement that, *if u is an interpoint, there exists a point containing u.* In symbols,

$$\text{VII Hp R} \,.=:\, (t, u) : u \,\epsilon\, \text{intpnt}_{\text{R}t} \,.\supset.\, (\exists p) \,.\, p \,\epsilon\, \text{pnt}_{\text{R}t} \,.\, u \subset p \qquad \text{Df}$$

*22·33. VIII Hp R is the statement that, *if p be a point, and u and v be two distinct interpoints contained in p, then u and v possess no common member.* In symbols,

$$\text{VIII Hp R} \,.=:\, (p, u, v, t) : p \,\epsilon\, \text{pnt}_{\text{R}t} \,.\, u, v \,\epsilon\, \text{intpnt}_{\text{R}t} \cap \text{cls}`p \,.\, u \neq v \,.\supset.\, u \cap v = \Lambda \quad \text{Df}$$

The next set of three axioms, viz. (IX.–XI.) Hp R, supplies the missing hypotheses requisite to make homaloty a "geometrical property," as defined in *10.

*22·41. IX Hp R is the statement that, *if t is an instant of time, ν Hp $\mu_{\text{R}t}$ is true.* In symbols,

$$\text{IX Hp R} \,.=:\, t \,\epsilon\, \text{T} \,.\supset_t.\, \nu \text{ Hp } \mu_{\text{R}t} \qquad \text{Df } (cf. \text{ *10·3})$$

*22·42. X Hp R is the statement that, *if t is an instant of time, π Hp $\mu_{\text{R}t}$ is true.* In symbols,

$$\text{X Hp R} \,.=:\, t \,\epsilon\, \text{T} \,.\supset_t.\, \pi \text{ Hp } \mu_{\text{R}t} \qquad \text{Df } (cf. \text{ *10·4})$$

*22·43. XI Hp R is the statement that, *if t is an instant of time, ρ Hp $\mu_{\text{R}t}$ is true.* In symbols,

$$\text{XI Hp R} \,.=:\, t \,\epsilon\, \text{T} \,.\supset_t.\, \rho \text{ Hp } \mu_{\text{R}t} \qquad \text{Df } (cf. \text{ *10·5})$$

*22·51. XII Hp R is the statement that, *if p and q are distinct planes, and there exists a point, not a cogredient point, which is a member of the punctual associates of both planes, then p and q possess a common member.* In symbols,

$$\text{XII Hp R} \,.=:\, p, q \,\epsilon\, \text{ple}_{\text{R}t} \,.\, p \neq q \,.\, \exists ! \{(\text{ass}_{\text{R}t}`p \cap \text{ass}_{\text{R}t}`q) - \infty_{\text{R}t}\} \,.\supset_{p,q,t}.\, \exists ! (p \cap q) \quad \text{Df}$$

The next axiom, XIII Hp R, is the "Euclidean" axiom.

*22·61. XIII Hp R is the statement that *the cogredient points are points.* In symbols,

$$\text{XIII Hp R} \,.=.\, \infty_{\text{R}t} \subset \text{pnt}_{\text{R}t} \qquad \text{Df}$$

The next three axioms, namely (XIV.–XVI.) Hp R, establish the theory of the order of points as determined by the point-ordering relation (*cf.* *20·51). Incidentally some existence theorems can be deduced from them, which would else have to be provided for elsewhere.

*22·71. XIV Hp R is the statement that, *if* A *and* B *are two distinct non-*

cogredient points, then there exists at least one point C *such that* A, B, C *are in the point-order* ABC *at the instant considered.* In symbols,

$$\text{XIV Hp R} . = : (A, B, t) : A, B \epsilon \text{pnt}_{Rt} - \infty_{Rt} . \supset . \exists ! R_{pn}`(AB;t) \qquad \text{Df}$$

*22·72. XV Hp R is the statement that, *if* A, B, C *are three distinct non-cogredient points, on the same objective real, then at the instant considered one of t point-orders* ABC, *or* BCA, *or* CAB *holds.* In symbols,

$$\text{XV Hp R} . = .\therefore (A, B, C, t) .\therefore A, B, C \epsilon \text{pnt}_{Rt} - \infty_{Rt} . \exists ! (A \cap B \cap C) .$$
$$A \neq B . B \neq C . C \neq A . \supset : R_{pn}`(ABCt) . \vee . R_{pn}`(BCAt) . \vee . R_{pn}`(CABt) \qquad \text{Df}$$

The next axiom, XVI Hp R, is the well-known "transversal" axiom.

*22·73. XVI Hp R is the statement that, *if at the instant t the points* B, C, D *are in the point-order* BCD, *and the points* C, E, A *are in the point-order* CEA, *and the points* A, B, C *are not collinear, and* F *lies in the punctual associates both of* $A \cap B$ *and of* $D \cap E$, *then the points* A, F, B *are in the point-order* AFB. In symbols,

$$\text{XVI Hp R} . = : (A, B, C, D, E, F, t) : R_{pn}`(BCDt) . R_{pn}`(CEAt) .$$
$$A \cap B \cap C = \Lambda . F \epsilon \text{ass}_{Rt}`(A \cap B) \cap \text{ass}_{Rt}`(D \cap E) . \supset . R_{pn}`(AFBt) \qquad \text{Df}$$

*22·74. As XVII Hp R, an axiom of continuity will be wanted.

Note.—The above axioms are all axioms of *geometry*, in the sense of "geometry" as defined in the sense definition of it given in Part I. (i.). But geometry in this Concept V. includes more than does geometry in Concept I. For in Concept I. geometry has only to do with points, punctual lines, and punctual planes; but in Concept V. geometry has, in addition, to consider the relation of the objective reals (which are all "linear") and of interpoints to the above entities. In this respect, geometry in Concept V. merges into physics more than does geometry in Concept I. Thus the excess of the number of axioms in Concept V. over the number in Concept I. arises from the fact that there is a larger field to be covered. Also, I Hp R is not required in the geometrical reasoning.

*25. *Preliminary Propositions.*

*25·11. *Proposition.*—Assuming (II.–VI) Hp R, all the propositions of the theory of interpoints (*cf.* *1) hold of the interpoints of this Concept.

*25·12. *Proposition.*—Assuming (II.–VI.) Hp R, if t be an instant of time, then O_{Rt} possesses at least *four* members. In symbols,

$$\vdash .\therefore \text{IX Hp R} . \supset :. t \epsilon T . \supset . Nc`O_{Rt} \geqq 4$$

Proof.—*Cf.* *1·61·71·72·73 and *21·11 and *25·11.

*25·13. *Proposition.*—Assuming (II.–VI.) $\mathrm{H}p\,\mathrm{R}$, if t be an instant of time, then $\mathrm{O}_{\mathrm{R}t}$ is homalous. In symbols,

$$\vdash .\because (\mathrm{II.–VI.})\,\mathrm{H}p\,\mathrm{R} \,.\supset :\, t \in \mathrm{T} \,.\supset .\, \mu_{\mathrm{R}t}!\mathrm{O}_{\mathrm{R}t}$$

Proof.—From *1·65 and *21·11, $\mathrm{O}_{\mathrm{R}t}$ is identical with $\mathrm{R}^{\prime}(;\cdots t)$. Hence (*cf.* *1·31·61·62) every member of $\mathrm{O}_{\mathrm{R}t}$ is doubly secant with $\mathrm{O}_{\mathrm{R}t}$. Again (*cf.* *21·11), every objective real which is doubly secant with $\mathrm{O}_{\mathrm{R}t}$ is a member of it.

*25·14. *Proposition.*—Assuming (II.–VI., IX–XI.) $\mathrm{H}p\,\mathrm{R}$, if t be an instant of time, then all the special deductions of the theory of dimensions, namely *11 to *16, hold respecting homaloty, that is, with $\mu_{\mathrm{R}t}$ substituted for ϕ.

Proof.—*Cf.* *21·31 and *22·41·42·43 and *25·13.

*25·21. *Proposition.*—Assuming I $\mathrm{H}p\,\mathrm{R}$, if t is an instant of time, then $\mathrm{O} = \mathrm{O}_{\mathrm{R}t}$. In symbols,

$$\vdash .\because \mathrm{I\,H}p\,\mathrm{R} \,.\supset :\, t \in \mathrm{T} \,.\supset .\, \mathrm{O} = \mathrm{O}_{\mathrm{R}t}.$$

Proof.—*Cf.* *21·11 and *22·1.

Note.—The above theorem is not used in any geometrical reasoning.

*25·31. *Proposition.*—Assuming (II.–VIII.) $\mathrm{H}p\,\mathrm{R}$, if a be a member of $\mathrm{O}_{\mathrm{R}t}$, then the number of points on a is at least three. In symbols,

$$\vdash .\because (\mathrm{II.–VIII.})\,\mathrm{H}p\,\mathrm{R} \,.\supset :\, a \in \mathrm{O}_{\mathrm{R}t} \,.\supset .\, \mathrm{N}c\text{‘}\mathrm{ass}_{\mathrm{R}t}\text{‘}\iota\text{‘}a \geqq 3$$

Proof.—*Cf.* *1·72 and *21·11 and *22·32·33 and *25·1.

*25·32. *Proposition.*—Assuming (II.–VII., IX.–XI.) $\mathrm{H}p\,\mathrm{R}$, if u be an interpoint, there is one and only one point containing it. In symbols,

$$\vdash .\because (\mathrm{II.–VII.,\ IX.–XI.})\,\mathrm{H}p\,\mathrm{R} \,.\supset :\, u \in \mathrm{intpnt}_{\mathrm{R}t} \,.\supset .\, \hat{\mathrm{P}}\,\{\mathrm{P} \in \mathrm{pnt}_{\mathrm{R}t} \,.\, u \subset \mathrm{P}\} \in 1$$

Proof.—*Cf.* *1·71 and *14·21 and *22·32.

*26. *On Cogredient Points.*

*26·11. *Proposition.*—Assuming (II.–VI., IX–XI., XIII.) $\mathrm{H}p\,\mathrm{R}$, if a point possesses two members which are cogredient to each other, it is a cogredient point. In symbols,

$$\vdash .\because (\mathrm{II.–VI.,\ IX.–XI.,\ XIII.})\,\mathrm{H}p\,\mathrm{R} \,.\supset :$$
$$\mathrm{A} \in \mathrm{pnt}_{\mathrm{R}t} \,.\, a, b \in \mathrm{A} \,.\, b \in \mathrm{cogrd}_{\mathrm{R}t}\text{‘}a \,.\, a \neq b \,.\supset .\, \mathrm{A} \in \infty_{\mathrm{R}t}$$

Proof.—*Cf.* *14·21 and *20·42 and *22·61.

*26·22. *Proposition.*—Assuming (II.–VI., IX.–XI., XIII.) $\mathrm{H}p\,\mathrm{R}$, if A is a cogredient point and a is a member of A, then A is identical with $\iota\text{‘}a \cup \mathrm{cogrd}_{\mathrm{R}t}\text{‘}a$. In symbols,

$$\vdash .\because (\mathrm{II.–VI.,\ IX.–XI.,\ XIII.})\,\mathrm{H}p\,\mathrm{R} \,. \;:\, \mathrm{A} \in \infty_{\mathrm{R}t} \,.\, a \in \mathrm{A} \,.\supset .\, \mathrm{A} = \iota\text{‘}a \cup \mathrm{cogrd}_{\mathrm{R}t}\text{‘}a$$

Proof.—*Cf.* *14·21 and *20·42 and *21·41 and *22·61.

*26·23. *Proposition.*—Assuming (II.–VI., IX.–XI., XIII.) Hp R, there is one and only one cogredient point lying in the punctual associate of an objective real. In symbols,

$$\vdash .\therefore (\text{II.–VI., IX.–XI., XIII.})\ \text{H}p\ \text{R} . \supset : a \in \text{O}_{\text{R}t} . \supset . \infty_{\text{R}t} \cap \text{ass}_{\text{R}t}`\iota`a \in 1$$

Proof.—*Cf.* *26·22.

*26·24. *Proposition.*—Assuming (II.–XI., XIII.) Hp R, there are at least two points, not cogredient points, lying in the punctual associate of an objective real. In symbols,

$$\vdash .\therefore (\text{II.–XI., XIII.})\ \text{H}p\ \text{R} . \supset : a \in \text{O}_{\text{R}t} . \supset . Nc`\{\text{ass}_{\text{R}t}`\iota`a - \infty_{\text{R}t}\} \geqq 2$$

Proof.—*Cf.* *25·31 and *26·23.

*27. *On Punctual Lines.*

*27·11. *Proposition.*—Assuming (II.–VI., IX.–XI.) Hp R, if p and q are distinct planes, and $p \cap q$ possesses a member, then $\text{ass}_{\text{R}t}`p \cap \text{ass}_{\text{R}t}`q$ is identical with $\text{ass}_{\text{R}t}`(p \cap q)$. In symbols,

$$\vdash .\therefore (\text{II.–VI., IX.–XI.})\ \text{H}p\ \text{R} . \supset :$$
$$p, q \in \text{ple}_{\text{R}t} . p \neq q . \exists!(p \cap q) . \supset . \text{ass}_{\text{R}t}`p \cap \text{ass}_{\text{R}t}`q = \text{ass}_{\text{R}t}`(p \cap q)$$

Proof.—*Cf.* *16·33 and *20·21.

*27·12. *Proposition.*—Assuming (II.–VI., IX.–XII.) Hp R, if p and q are distinct planes, and a point, not a cogredient point, lies in the punctual associates both of p and also of q, then $p \cap q$ possesses one and only one member. In symbols,

$$\vdash .\therefore (\text{II.–VI., IX.–XII.})\ \text{H}p\ \text{R} . \supset :$$
$$p, q \in \text{ple}_{\text{R}t} . p \neq q . \exists!\{(\text{ass}_{\text{R}t}`p \cap \text{ass}_{\text{R}t}`q) - \infty_{\text{R}t}\} . \supset . p \cap q \in 1$$

Proof.—*Cf.* *16·21 and *22·51.

*27·13. *Proposition.*—Assuming (II.–XIII.) Hp R, if p and q are distinct planes, then if $p \cap q$ possesses one member, there are non-cogredient points lying in the punctual associates both of p and of q; and also conversely. In symbols,

$$\vdash :: (\text{II.–XIII.})\ \text{H}p\ \text{R} . \supset .\therefore p, q \in \text{ple}_{\text{R}t} . p \neq q . \supset :$$
$$\exists!\{(\text{ass}_{\text{R}t}`p \cap \text{ass}_{\text{R}t}`q) - \infty_{\text{R}t}\} . \equiv . p \cap q \in 1$$

Proof.—*Cf.* *26·24 and *27·12.

*27·21. *Proposition.*—Assuming (II.–VI., IX.–XII.) Hp R, if m is a punctual line possessing a non-cogredient point, then there exists an objective real such that m is its punctual associate. In symbols,

$$\vdash .\therefore (\text{II.–VI., IX.–XII.})\ \text{H}p\ \text{R} . \supset :$$
$$m \in \text{lin}_{\text{R}t} . \exists!(m - \infty_{\text{R}t}) . \supset . (\exists a) . a \in \text{O}_{\text{R}t} . m = \text{ass}_{\text{R}t}`\iota`a$$

Proof.—*Cf.* *27·11·12.

*27·22. *Proposition.*—Assuming (II.–XIII.) Hp R, a punctual line possesses either more than one non-cogredient point or no such point. In symbols,

$$\vdash .\therefore (\text{II.–XIII.})\, \text{H}p\, \text{R} \,.\, \supset : m \,\epsilon\, \text{lin}_{\text{R}t} \,.\, \exists\, ! \{m - \infty_{\text{R}t}\} \,.\, \supset .\, \text{N}c`(m - \infty_{\text{R}t}) > 1$$

Proof.—*Cf.* *20·22 and *26·24 and *27·21.

*27·23. *Proposition.*—Assuming (II.–VI., IX.–XIII.) Hp R, a punctual line, possessing a non-cogredient point, possesses one and only one cogredient point. In symbols,

$$\vdash .\therefore (\text{II.–VI., IX.–XIII.})\, \text{H}p\, \text{R} \,.\, \supset : m \,\epsilon\, \text{lin}_{\text{R}t} \,.\, \exists\, ! (m - \infty_{\text{R}t}) \,.\, \supset .\, m \cap \infty_{\text{R}t} \,\epsilon\, 1$$

Proof.—*Cf.* *26·23 and *27·21.

*27·31. *Proposition.*—Assuming (II.–VI., IX.–XII.) Hp R, if m is a punctual line possessing a non-cogredient point, and A and B are two distinct points lying in it, then m is the punctual associate of A ∩ B. In symbols,

$$\vdash .\therefore (\text{II.–VI., IX.–XII.})\, \text{H}p\, \text{R} \,.\, \supset :$$
$$m \,\epsilon\, \text{lin}_{\text{R}t} \,.\, \exists\, ! (m - \infty_{\text{R}t}) \,.\, \text{A, B} \,\epsilon\, m \,.\, \text{A} \neq \text{B} \,.\, \supset .\, m \,\epsilon\, \text{ass}_{\text{R}t}`(\text{A} \cap \text{B})$$

Proof.—*Cf.* *14·21 and *27·11·12.

*27·41. *Proposition.*—Assuming (II.–XI.) Hp R, if a is any objective real, then there exist two planes p and q such that a is the sole member of $p \cap q$. In symbols,

$$\vdash .\therefore (\text{II.–XI.})\, \text{H}p\, \text{R} \,.\, \supset : a \,\epsilon\, \text{O}_{\text{R}t} \,.\, \supset .\, (\exists p, q) \,.\, p, q \,\epsilon\, \text{ple}_{\text{R}t} \,.\, p \cap q = \iota`a$$

Proof.—*Cf.* *12·21 and *14·12·14 and *22·41 and *25·31.

*27·42. *Proposition.*—Assuming (II.–XIII.) Hp R, if a be an objective real, then $\text{ass}_{\text{R}t}`\iota`a$ is a punctual line with a non-cogredient point, and conversely, if m is a punctual line with a non-cogredient point, there exists an objective real a such that $m = \text{ass}_{\text{R}t}`\iota`a$. In symbols,

$$\vdash .\therefore (\text{II.–XIII.})\, \text{H}p\, \text{R} \,.\, \supset : m \,\epsilon\, \text{lin}_{\text{R}t} \,.\, \exists\, ! (m - \infty_{\text{R}t}) \,.\, \equiv .\, (\exists a) \,.\, a \,\epsilon\, \text{O}_{\text{R}t} \,.\, m = \text{ass}_{\text{R}t}`\iota`a$$

Proof.—*Cf.* *27·11·13·21·41.

Note.—This proposition, *27·42, establishes the connection between the objective reals and the punctual lines.

*27·43. *Proposition.*—Assuming (II.–XIII.) Hp R, if a and c are cogredient, they are distinct. In symbols,

$$\vdash .\therefore (\text{II.–XIII.})\, \text{H}p\, \text{R} \,.\, \supset : c \,\epsilon\, \text{cogrd}_{\text{R}t}`a \,.\, \supset .\, a \neq c$$

Proof.—*Cf.* *20·31·41 and *27·42.

*27·51. *Proposition.*—Assuming (II.–XIV.) Hp R, if A is any non-cogredient point

and B is any other point, then A $\cap$ B possesses one and only one member. In symbols,

$$\vdash \therefore (\text{II.–XIV.})\, \mathrm{H}p\, \mathrm{R} \,.\supset: \mathrm{A} \epsilon \mathrm{pnt}_{\mathrm{R}t} - \infty_{\mathrm{R}t} \,.\, \mathrm{B} \epsilon \mathrm{pnt}_{\mathrm{R}t} \,.\, \mathrm{A} \neq \mathrm{B} \,.\supset.\, \mathrm{A} \cap \mathrm{B} \epsilon 1$$

Proof.—(i) If B is non-cogredient, *cf.* *14·21 and *20·51 and *22·71. (ii) If B is cogredient, then (*cf.* *14·14) let b be a member of B. Then (*cf.* *26·24) there is on b a non-cogredient point D. Hence by (i) A $\cap$ D possesses a single objective real, d say. Hence (*cf.* *14·12) b and d are coplanar and $\mathrm{cm}_{\mathrm{R}t}$‘$(\iota$‘$b \cup \iota$‘$d)$ is a plane whose punctual associate possesses both A and B. Also, since a point is three-dimensional, B possesses another objective real, c say, not coplanar with b and d. Hence by similar reasoning $\mathrm{cm}_{\mathrm{R}t}$‘$(\iota$‘$b \cup \iota$‘$c)$ is a plane, not identical with $\mathrm{cm}_{\mathrm{R}t}$‘$(\iota$‘$b \cup \iota$‘$d)$, whose punctual associate also possesses A and B. Hence (*cf.* *27·13) these two planes have one objective real in common, and hence (*cf.* *16·33) this objective real is a member of A $\cap$ B, and hence (*cf.* *14·21) A $\cap$ B possesses one and only one member.

*27·52. *Proposition.*—Assuming (II.–XIV.) Hp R, if A be a non-cogredient point and B be any other point, then $\mathrm{ass}_{\mathrm{R}t}$‘(A $\cap$ B) is a punctual line with a non-cogredient point, and conversely, if m be a punctual line with a non-cogredient point, then there exist two points A and B, such that A is not cogredient and m is identical with $\mathrm{ass}_{\mathrm{R}t}$‘(A $\cap$ B). In symbols,

$$\vdash \therefore (\text{II.–XIV.})\, \mathrm{H}p\, \mathrm{R} \,.\supset: m \epsilon \mathrm{lin}_{\mathrm{R}t} \,.\, \exists!(m - \infty_{\mathrm{R}t}) \,.\equiv.$$
$$(\exists \mathrm{A}, \mathrm{B}) \,.\, \mathrm{A} \epsilon \mathrm{pnt}_{\mathrm{R}t} - \infty_{\mathrm{R}t} \,.\, \mathrm{B} \epsilon \mathrm{pnt}_{\mathrm{R}t} \,.\, \mathrm{A} \neq \mathrm{B} \,.\, m = \mathrm{ass}_{\mathrm{R}t}‘(\mathrm{A} \cap \mathrm{B})$$

Proof.—*Cf.* *27·22·31·42·51.

*28. *On Figures.*

*28·01. *Proposition.*—Assuming (II.–VI., IX.–XI.) Hp R, if t be an instant of time, there exists at least one punctual plane, not the plane $\infty_{\mathrm{R}t}$. In symbols,

$$\vdash \therefore (\text{II.–VI., IX.–XI.})\, \mathrm{H}p\, \mathrm{R} \,.\supset: t \epsilon \mathrm{T} \,.\supset.\, \exists!(\mathrm{pple}_{\mathrm{R}t} - \iota‘\infty_{\mathrm{R}t})$$

Proof.—*Cf.* *12·42 and *22·41.

*28·11. *Proposition.*—Assuming (II.–XIII.) Hp R, if p be any punctual plane, not the plane $\infty_{\mathrm{R}t}$, it possesses at least three non-cogredient points, which are not collinear. In symbols,

$$\vdash \therefore (\text{II.–XIII.})\, \mathrm{H}p\, \mathrm{R} \,.\supset: p \epsilon \mathrm{pple}_{\mathrm{R}t} - \iota‘\infty_{\mathrm{R}t} \,.\supset.$$
$$(\exists u) \,.\, u \epsilon (3 \cap \mathrm{cls}‘p).\, u \cap \infty_{\mathrm{R}t} = \Lambda \,.\sim \mathrm{coll}_{\mathrm{R}t}!u$$

Proof.—*Cf.* *14·21 and *26·24 and *27·21·31.

Note.—*Cf.* *16·42 and the note on it.

*28·12. *Proposition.*—Assuming (II.–XIII.) Hp R, if p be any punctual plane, not

the plane ∞_{Rt}, there exists at least one non-cogredient point not lying in it. In symbols,

$$\vdash .\because (\text{II.–XIII.})\ \text{H}p\ \text{R} . \supset : p \in \text{pple}_{Rt} - \iota ' \infty_{Rt} . \supset . (\exists \text{A}) . \text{A} \in (\text{pnt}_{Rt} - \infty_{Rt}) - p$$

Proof.—*Cf.* *12·21 and *16·32 and *22·41 and *26·24 and *28·11.

*28·21. *Proposition.*—Assuming (II.–XIV.) Hp R, if x be any objective real and p be any plane, then either the punctual associates of x and p have one and only one common member, or x is a member of p. In symbols,

$$\vdash :: (\text{II.–XIV.})\ \text{H}p\ \text{R} . \supset .\because x \in \text{O}_{Rt} . p \in \text{ple}_{Rt} . \supset : \text{ass}_{Rt} ' \iota ' x \cap \text{ass}_{Rt} ' p \in 1 . \vee . x \in p$$

Proof.—Take (*cf.* *26·24 and *28·11) two non-cogredient points A and B upon x, and a non-cogredient point C in $\text{ass}_{Rt}'p$ but not on x. If either A or B lie in $\text{ass}_{Rt}'p$, then *cf.* *16·32. If neither A nor B lie in $\text{ass}_{Rt}'p$, then (*cf.* *16·32 and *27·51) $\text{cm}_{Rt}'\{(\text{B} \cap \text{C}) \cup (\text{A} \cap \text{C})\}$ is a plane possessing x, and its punctual associate possesses C. Hence (*cf.* *22·51) there is a common member of p and this plane, y say, and x and y are coplanar, hence (*cf.* *10·4 and *16·11) the punctual associates of x and y possess a common point. Hence $\text{ass}_{Rt}'\iota'x$ and $\text{ass}_{Rt}'p$ possess a point in common, and then *cf.* *16·32.

*28·22. *Proposition.*—Assuming (II.–XIV.) Hp R, if p and q are punctual planes, and p is not identical with ∞_{Rt}, and $p \cap q$ is contained in ∞_{Rt}, then $p \cap q$ and $p \cap \infty_{Rt}$ are identical. In symbols,

$$\vdash .\because (\text{II.–XIV.})\ \text{H}p\ \text{R} . \supset : p, q \in \text{pple}_{Rt} . p \neq \infty_{Rt} . p \cap q \subset \infty_{Rt} . \supset . p \cap q = p \cap \infty_{Rt}$$

Proof.—If p and q are identical, then $p \subset \infty_{Rt}$, but (*cf.* *28·11) this is impossible. Hence $p \neq q$. If $q = \infty_{Rt}$, then $p \cap q = p \cap \infty_{Rt}$. Assume $q \neq \infty_{Rt}$. Then (*cf.* *20·61) there exist planes, p' and q' say, such that p is $\text{ass}_{Rt}'p'$ and $q = \text{ass}_{Rt}'q'$. Since $p \cap q \subset \infty_{Rt}$, there is (*cf.* *26·24) no objective real common to p' and q'. Hence (*cf.* *28·21) upon every objective real possessed by p' there is one and only one point lying in q. Hence (*cf.* *26·23) $p \cap q = p \cap \infty_{Rt}$.

*28·31. *Proposition.*—Assuming (II.–XIV.) Hp R, in every punctual line there lie at least three points. In symbols,

$$\vdash .\because (\text{II.–XIV.})\ \text{H}p\ \text{R} . \supset : m \in \text{lin}_{Rt} . \supset . \text{N}c'm \geqq 3$$

Proof.—If m possesses non-cogredient points, then *cf.* *27·22·23. If m is contained in ∞_{Rt}, then (*cf.* *28·22) m is identical with $p \cap \infty_{Rt}$, where p is a punctual plane. But (*cf.* *27·51·52 and *28·11) there are three distinct punctual lines contained in p, meeting two by two in three non-cogredient points; then *cf.* *27·23.

*28·32. *Proposition.*—Assuming (II.–XIV.) Hp R, a punctual line is the common meeting of two punctual planes, and conversely. In symbols,

$$\vdash .\because (\text{II.–XIV.})\ \text{H}p\ \text{R} . \supset : \text{lin}_{Rt} = \hat{m}\{(\exists p, q) . p, q \in \text{pple}_{Rt} . p \neq q . m = p \cap q\}$$

Proof.—The direct proposition follows from *20·22·61. For the converse, let p and q be a pair of punctual planes. If neither p nor q be ∞_{Rt}, then *cf.* *20·22·61. Consider now $p \cap \infty_{Rt}$, where p is distinct from ∞_{Rt}. Take (*cf.* *28·12) a non-cogredient point A, not in p. Also (*cf. proof* of *28·31) there are two distinct points B and C in $p \cap \infty_{Rt}$. Hence (*cf.* *27·51) $\mathrm{cm}_{Rt}`\{(A \cap B) \cup (A \cap C)\}$ is a plane, and its punctual associate, q say, possesses A and B and is distinct from p. Hence (*cf.* *27·23 and *28·22) $p \cap q$ is identical with $p \cap \infty_{Rt}$. But (*cf.* *28·22) $p \cap q$ is a punctual line, and hence $p \cap \infty_{Rt}$ is a punctual line.

*28·33. *Proposition.*—Assuming (II.–XIV.) Hp R, two distinct points lie in one and only one punctual line. In symbols,

$$\vdash .:. (\text{II.–XIV.})\, \mathrm{H}p\, \mathrm{R} \,.\supset :\, \mathrm{A}, \mathrm{B} \in \mathrm{pnt}_{Rt} \,.\, \mathrm{A} \neq \mathrm{B} \,.\supset .\, m\{m \in \mathrm{lin}_{Rt} \,.\, \mathrm{A}, \mathrm{B} \in m\} \in 1$$

Proof.—Firstly, only one punctual line (if any) possesses both A and B (*cf.* *27·31 and *28·22·32). Secondly, to prove that a punctual line exists possessing both A and B. If either point is non-cogredient, *cf.* *27·52. If both points are cogredient, then (*cf.* *28·11·12) two non-cogredient points C and D exist such that the four points A, B, C, D are not punctually coplanar. Hence (*cf.* *27·51) the meeting of the punctual associates of $\mathrm{cm}_{Rt}`\{(A \cap C) \cup (B \cap C)\}$ and of $\mathrm{cm}_{Rt}`\{(A \cap D) \cup (B \cap D)\}$ is a punctual line possessing A and B.

*28·41. *Proposition.*—Assuming (II.–XIV.) Hp R, three points, which are not collinear, lie in one and only one plane. In symbols,

$$\vdash .:. (\text{II.–XIV.})\, \mathrm{H}p\, \mathrm{R} \,.\supset :\, u \in 3 \cap \mathrm{pnt}_{Rt} \,.\, \backsim \mathrm{coll}_{Rt}!u \,.\supset .\, \dot{p}\{p \in \mathrm{pple}_{Rt} \,.\, u \subset p\} \in 1$$

Proof.—If the three points are all cogredient, then (*cf.* *28·22·32) ∞_{Rt} is the only punctual plane which possesses them all. If the three points are A, B, C, and A be non-cogredient, then (*cf.* *27·51 and *28·32) the punctual associate of $\mathrm{cm}_{Rt}`\{(A \cap B) \cup (A \cap C)\}$ is a punctual plane, possessing A, B, and C, and is the only one.

*28·42. *Proposition.*—Assuming (II.–XIV.) Hp R, three punctual planes, which do not meet in a punctual line, meet in one point. In symbols,

$$\vdash .:. (\text{II.–XIV.})\, \mathrm{H}p\, \mathrm{R} \,.\supset :\, u \in 3 \cap \mathrm{cls}`\mathrm{pple}_{Rt} \,.\supset .\, \cap `u \in \mathrm{lin}_{Rt} \cup 1$$

Proof.—Let p, q, r be the three punctual planes. Assume that p and q are neither the punctual plane ∞_{Rt}. If $q \cap r$ is contained in ∞_{Rt}, then *cf.* *20·22·61 and *27·23 and *28·22·32. If $q \cap r$ is not contained in ∞_{Rt}, then *cf.* *27·12 and *28·21.

*30. *Perspective.*

A few propositions on perspective (*cf.* *20·31·32·33) are required as a preliminary to the discussion of the point-ordering relation (*cf.* *20·51).

*30·1. *Proposition.*—Assuming (II.–XIV.) $\mathrm{H}p\,\mathrm{R}$, if two figures are in perspective, their cardinal numbers are equal and each greater than one. In symbols,

$$\vdash .\therefore (\text{II.–XIV.})\ \mathrm{H}p\,\mathrm{R} \,.\supset :\, u \operatorname{persp}_{\mathrm{R}t} v \,.\supset .\, \mathrm{N}c`u = \mathrm{N}c`v \,.\, \mathrm{N}c`u > 1$$

Proof.—The equality of the cardinal numbers follows from the definition; also if both figures were unit classes, then (*cf.* *28·33) they would be collinear.

*30·3. *Proposition.*—Assuming (II.–XIV.) $\mathrm{H}p\,\mathrm{R}$, if the figure u is in perspective with the figure v, and also with the figure w, and if u, v, w are respectively collinear, and the punctual lines respectively containing u, v, w possess a common meeting, then either v is in perspective with w, or the joint class of v and w (*i.e.*, $v \cup w$) is collinear. In symbols,

$$\vdash :: (\text{II.–XIV.})\ \mathrm{H}p\,\mathrm{R} \,.\supset .\therefore\, u \operatorname{persp}_{\mathrm{R}t} v \,.\, u \operatorname{persp}_{\mathrm{R}t} v \,.\, m, m', m'' \in \mathrm{lin}_{\mathrm{R}t} \,.$$
$$u \subset m \,.\, v \subset m' \,.\, w \subset m'' \,.\, \exists!(m \cap m' \cap m'') \,.\supset :\, v \operatorname{persp}_{\mathrm{R}t} w \,.\vee .\, m' = m''$$

Proof.—DESARGUES' well-known propositions respecting triangles in perspective being coaxial, and its converse, can now (*cf.* *28·11·12·31·32·33·41·42) be proved. Then by drawing a figure for the present proposition the conclusion easily follows from some pure geometrical reasoning.

*31. *The Point-Ordering Relation.*

It will be proved in this section that the point-ordering relation (R_{pn}) has at any instant the same properties as the essential relation of Concept I. (*cf.* *31·3). It follows that the ordinary Euclidean geometry holds of the figures of Concept V., the points at infinity being the points of the punctual plane $\infty_{\mathrm{R}t}$, and the metrical ideas being introduced by appropriate definitions.

*31·11. *Proposition.*—Assuming (II.–XI., XIII., XIV.) $\mathrm{H}p\,\mathrm{R}$, the class $\mathrm{R}_{\mathrm{pn}}{}^{;}(;;;t)$ is identical with the class of non-cogredient points. In symbols,

$$\vdash .\therefore (\text{II.–XI., XIII., XIV.})\ \mathrm{H}p\,\mathrm{R} \,.\supset .\, \mathrm{R}_{\mathrm{pn}}{}^{;}(;;;t) = \mathrm{pnt}_{\mathrm{R}t} - \infty_{\mathrm{R}t}$$

Proof.—*Cf.* *3·42 and *20·51 and *22·71 and *26·24.

*31·12. *Proposition.*—Assuming (II.–XV.) $\mathrm{H}p\,\mathrm{R}$, if a is a punctual line, and A and B are two non-cogredient points on it, then a, without its cogredient point, is identical with the whole class formed by $\mathrm{R}_{\mathrm{pn}}{}^{;}(;\mathrm{AB}t)$ and $\mathrm{R}_{\mathrm{pn}}{}^{;}(\mathrm{A};\mathrm{B}t)$ and $\mathrm{R}^{;}(\mathrm{AB};t)$ together with A and B added as members. In symbols,

$$\vdash .\therefore (\text{II.–XV.})\ \mathrm{H}p\,\mathrm{R} \,.\supset :\, a \in \mathrm{lin}_{\mathrm{R}t} \,.\, \mathrm{A}, \mathrm{B} \in a - \infty_{\mathrm{R}t} \,.\, \mathrm{A} \neq \mathrm{B} \,.\supset .$$
$$a - \infty_{\mathrm{R}t} = \mathrm{R}_{\mathrm{pn}}{}^{;}(;\mathrm{AB}t) \cup \mathrm{R}_{\mathrm{pn}}{}^{;}(\mathrm{A};\mathrm{B}t) \cup \mathrm{R}_{\mathrm{pn}}{}^{;}(\mathrm{AB};t) \cup \iota`\mathrm{A} \cup \iota`\mathrm{B}$$

Proof.—The identity is to be proved by showing that each class contains the other. For one half of the proof, *cf.* *22·72 and *27·21. For the other half, *cf.* *20·51 and *27·21·31.

*31·21. *Proposition.*—Assuming (II.–VI.) Hp R, the point-order ABC implies the point-order CBA. In symbols,

$$\vdash .\because (\text{II.–VI.})\,\text{H}p\,\text{R} \,.\supset :\, \text{R}_{\text{pn}}{}^{;}(\text{ABC}t) \,.\supset .\, \text{R}_{\text{pn}}{}^{;}(\text{CBA}t)$$

Proof.—*Cf.* *1·63 and *20·51.

*31·22. *Proposition.*—Assuming (II.–VII., IX.–XI.) Hp R, if A, B, C are in the point-order ABC, then A, B, and C are distinct. In symbols,

$$\vdash .\because (\text{II.–VII., IX.–XI.})\,\text{H}p\,\text{R} \,.\supset :\, \text{R}_{\text{pn}}{}^{;}(\text{ABC}t) \,.\supset .\, \text{A} \neq \text{B} \,.\, \text{A} \neq \text{C} \,.\, \text{B} \neq \text{C}$$

Proof.—There are (*cf.* *20·51) interpoints u, v, w on a common objective real a, such that $\text{R}_{\text{in}}{}^{;}(uvwt)$ and [ABC] $\text{persp}_{\text{R}t}\,[u_{\text{R}t}v_{\text{R}t}w_{\text{R}t}]$, where (*cf.* *20·23 and *25·32) $u_{\text{R}t}$, $v_{\text{R}t}$, $w_{\text{R}t}$ are the dominant points of u, v, w. But (*cf.* *1·62 and *22·33) $u_{\text{R}t}$, $v_{\text{R}t}$, $w_{\text{R}t}$ are distinct points. Hence (*cf.* *20·32) A, B, C are distinct points.

*31·23. *Proposition.*—Assuming (II.–XV.) Hp R, the point-order ABC is inconsistent with the point-order BCA. In symbols,

$$\vdash .\because (\text{II.–XV.})\,\text{H}p\,\text{R} \,.\supset :\, \text{R}_{\text{pn}}{}^{;}(\text{ABC}t) \,.\supset .\, \sim \text{R}{}^{;}(\text{BCA}t)$$

Proof.—Since this proof is long, the paragraphs will be numbered for reference by (i), (ii), &c., prefixed.

(i) If u, v, w are interpoints on the objective real a, and u', v', w' interpoints on the objective real a', and a and a' are cogredient, and $[u_{\text{R}t}v_{\text{R}t}w_{\text{R}t}]\,\text{persp}_{\text{R}t}\,[u'_{\text{R}t}v'_{\text{R}t}w'_{\text{R}t}]$, then (*cf.* *20·23·41 and *25·32) u, v, w and u', v', w' must agree in interpoint order, and (*cf.* *1·64) each set of interpoints has only one (if any) interpoint order (counting uvw and wvu as the same order), and (*cf.* *27·43) a and a' are distinct.

(ii) From *20·51, $\text{R}_{\text{pn}}{}^{;}(\text{ABC}t)$ implies (α) that A, B, C are on an object real d; (β) that there are interpoints u, v, w on an objective real a, cogredient with d; (γ) that $\text{R}_{\text{in}}{}^{;}(uvwt)$; ($\delta$) that [ABC] $\text{persp}_{\text{R}t}\,[u_{\text{R}t}v_{\text{R}t}w_{\text{R}t}]$; and ($\epsilon$) (*cf.* *27·43) that a and d are distinct.

(iii) Assume that $\text{R}_{\text{pn}}{}^{;}(\text{BCA}t)$ also holds. Then, in addition to the entities of (ii), there exist interpoints u', v', w' on an objective real a', satisfying all the conditions of (ii) without changes, except that (ii, γ) becomes $\text{R}_{\text{in}}{}^{;}(v'w'u't)$. This assumption (iii) will now be proved to be absurd.

(iv) From XIII Hp R (*cf.* *22·61) and *26·11·22, d, a, a' are copunctual and a and a' are either cogredient or identical.

(v) Hence (*cf.* *30·3) *either* (Case I.) $[u_{\text{R}t}v_{\text{R}t}w_{\text{R}t}]\,\text{persp}_{\text{R}t}\,[u'_{\text{R}t}v'_{\text{R}t}w'_{\text{R}t}]$ *or* (Case II.), a and a' are identical.

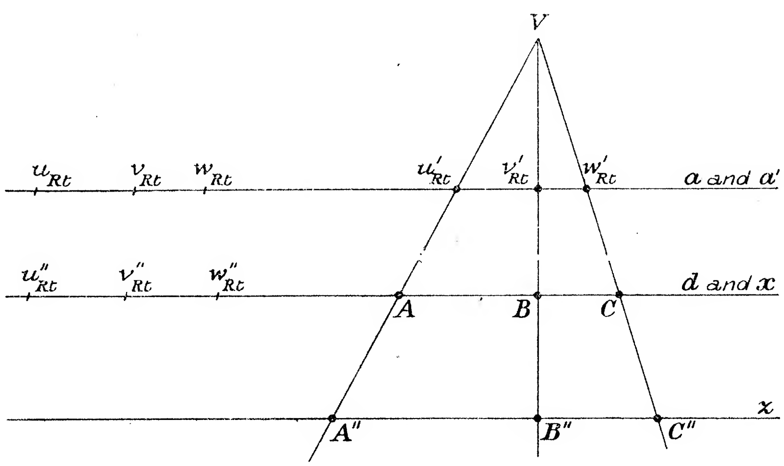
(vi). *Case I.*—We have $[u_{\text{R}t}v_{\text{R}t}w_{\text{R}t}]\,\text{persp}_{\text{R}t}\,[u'_{\text{R}t}v'_{\text{R}t}w'_{\text{R}t}]$. Hence (*cf.* *20·41) the interpoint orders of u, v, w and u', v', w' must agree. Hence, from (ii, γ), $\text{R}_{\text{in}}{}^{;}(u'v'w't)$ holds. But (*cf.* *1·64) this is inconsistent with $\text{R}_{\text{in}}{}^{;}(v'w'u't)$. Hence Case I. cannot hold.

(vii). *Case II.* —We have a and a' identical. Now (*cf.* *31·22) A, B, C are distinct points. Hence [*cf.* (ii, δ) and *30·1] u_{Rt}, v_{Rt}, w_{Rt} are distinct points, and [*cf.* (ii, β)] they are collinear. Hence, by XV Hp R (*cf.* *22·72), they are in some point-order. Hence [*cf.* (ii)] three interpoints u'', v'', w'' exist on a common objective real x, which is distinct from a and cogredient with it, and also $[u''_{Rt}v''_{Rt}w''_{Rt}]\,\text{persp}_{Rt}\,[u_{Rt}v_{Rt}w_{Rt}]$.

(viii). Hence Case II. divides into two subclasses, *either* (*Case II.*, α) x is not identical with d, *or* (*Case II.*, β) x is identical with d.

(ix). *Case II.*, α.—a and a' are identical and distinct from both d and x, and d and x are distinct; also a, d, x are cogredient and therefore (*cf.* *26·11·22) copunctual. Hence [*cf.* (ii, δ) and (vii) and *30·3] we have $[ABC]\,\text{persp}_{Rt}\,[u''_{Rt}v''_{Rt}w''_{Rt}]$. Hence [*cf.* (iii, δ) and *30·3] $[u'_{Rt}v'_{Rt}w'_{Rt}]\,\text{persp}_{Rt}\,[u''_{Rt}v''_{Rt}w''_{Rt}]$. Hence [*cf.* *20·41 and (ii, γ) and (vii)] $R_{in}{}^{;}(u'v'w't)$. Hence (*cf.* *1·64) Case II., α, cannot hold.

(x). *Case II.*, β.—a and a' are identical, x and d are identical, and a and d are cogredient and distinct. Since (*cf.* *20·51) A, B, C are not cogredient points, they are distinct from u'_{R}, v'_{R}, w'_{R}, since the only point common to the punctual associates of a and d is a cogredient point. Thus none of u'_{Rt}, v'_{Rt}, w'_{Rt} can be cogredient points. Hence (*cf.* *28·33) there is a punctual line joining B and w'_{Rt} which does not possess A or u'_{Rt} or V (*cf.* figure annexed). Hence (*cf.* *28·42) there is at least one other



point, A$''$ say, lying in the punctual line joining A and u'_{Rt} in addition to V and u'_{Rt} and A. Hence (*cf.* *28·33) there is a punctual line, z say, joining A$''$ to the cogredient point common to the punctual associates of a and d. This punctual line must meet (*cf.* *28·42) the punctual lines VB and VC (*cf.* figure annexed) in the points B$''$ and C$''$. Hence

$$[A''B''C'']\,\text{persp}_{Rt}\,[ABC],$$

and hence (*cf.* *30·3) we have

$$[A''B''C'']\,\text{persp}_{Rt}\,[u_{Rt}v_{Rt}w_{Rt}],$$

and hence (*cf.* *30·3) we have

$$[A''B''C'']\,\text{persp}_{Rt}\,[u''_{Rt}v''_{Rt}w''_{Rt}].$$

But also

$$[A''B''C''] \text{ persp}_{Rt} [u'_{Rt}v'_{Rt}w'_{Rt}],$$

and hence (*cf.* *30·3) we have

$$[u'_{Rt}v'_{Rt}w'_{Rt}] \text{ persp}_{Rt} [u''_{Rt}v''_{Rt}w''_{Rt}].$$

Thence, by the same reasoning as that in (ix) for Case II., α, it follows that Case II., β, cannot hold.

(xi). Hence neither Case I. nor Case II. can hold; and therefore the proposition follows.

*31·3. *Proposition.*—The point-ordering relation (R_{pn}) satisfies all the axioms satisfied by the essential relation of Concept III., except the axiom of persistence for that concept.

Proof.—In order to prove this, we make a comparison, as in Concept III., with the axioms of Concept I.

For I H*p* R of Concept I., *cf.* *25·12 and *26·24 and *31·11.
For II H*p* R of Concept I., *cf.* *31·21.
For III H*p* R of Concept I., *cf.* *31·23.
For IV H*p* R of Concept I., *cf.* *31·22.
For V H*p* R of Concept I., *cf.* XIV H*p* R (*22·71).
For VI H*p* R of Concept I., *cf.* *28·33 and *31·12.
For VII H*p* R of Concept I., *cf.* *28·01·11.
For VIII H*p* R of Concept I., *cf.* XVI H*p* R (*22·73).
For IX H*p* R of Concept I., *cf.* *28·12.
For X H*p* R of Concept I., *cf.* *28·42.
For XI H*p* R of Concept I., *cf.* *XVII H*p* R (*22·74).
For XII H*p* R of Concept I., *cf.* *26·23 and *28·33.

Note.—In order to complete this comparison, it must be noticed that it follows from *31·12 that the punctual line, with its cogredient point excepted, is the line as defined on the analogy of Concept I. Also, it follows, from *28·32 and *28·42 and the propositions of *31, that the punctual plane, with its cogredient points excepted, is the plane as defined on the analogy of Concept I. Then the transition to projective geometry is made, not by constructing a fresh type of points (the projective points), but simply by putting back the class (∞_{Rt}) of cogredient points. Metrical geometry is then constructed in the well-known way,† making the plane (∞_{Rt}) of cogredient points to be the plane at infinity.

The Extraneous Relation.—For the purpose of enabling velocity and acceleration to be measured, an extraneous relation is required, in all respects similar to those required in Concepts III. and IV., and the description already given need not be repeated.

* *Cf.* VEBLEN, *loc. cit.*, for a sketch of this method; also CLEBSCH and LINDEMANN, *loc. cit.*

The Corpuscles.—We may distinguish five types of points. A point of *Type* (1) contains no interpoints, and consists only of its nonsecant part (*cf.* *20·231). A point of *Type* (2) contains a single interpoint and no nonsecant part. Such a point is a single interpoint. A point of *Type* (3) contains a single interpoint together with a nonsecant part. A point of *Type* (4) contains many interpoints with no nonsecant part. A point of *Type* (5) contains many interpoints together with a nonsecant point.

We seem to be precluded from considering the "particles" to be stable points by the same difficulty as to the resulting permanence of collinearity, which was explained in considering the corpuscles of Concept IV. It is evident that at this stage many subdivisions of Concept V. are possible, in respect to the ideas which may be formed of the nature of the corpuscle. The following sketch of a possible development is given because of its superior simplicity, and also because of a certain consonance which it possesses with some modern physical ideas.

It is evident that volumes, in which, in some sense, there is an excess or a defect of interpoints, can be conceived as being charged with one or other of the two sorts of electricity. This idea is taken as the basis of the following brief outline of a possible development of the concept. Let the interpoints be identified with negative electricity and the nonsecant parts of points with positive electricity. A point of type (1) is a negative electron; a point of type (2) is a positive electron. The persistence of existence of an isolated electron of either type is to be *defined* by persistence of type and continuity of motion. If the electron is not isolated, consider, for example, a volume in which electrons of type (2) either compose all the points, or, at least, are everywhere dense. Then the persistence of such a collection of electrons must be considered as a whole, and is *defined*, as in the simpler case, by persistence of type and continuity of motion.

Three methods of procedure now suggest themselves, *either* (*Case I.*) to assume that the electrons consist of single points, so that a corpuscle is a volume containing a large finite number of points of type (2), and a small finite number of points of type (1), *or* (*Case II.*) to assume that a corpuscle is a volume in which points of type (2) are (at least) everywhere dense, and which contains a finite number of points of type (1), *or* (*Case III.*) to assume that an electron of either type is essentially a volume (possibly with internal boundaries) in which points of the appropriate type are at least everywhere dense. In Case III. a corpuscle will be a relatively large electron of type (2) containing within it a finite number of relatively small electrons of type (1). Case III. has the merit, such as it is, of making the "inverse square" law of electricity appear somewhat natural. The field of force "at a point" produced by an electron may be conceived as proportional to the number of objective reals shared in common by the point and the "electric points" in the electron, and also to the number of these electric points. The number of electric points would be measured by the mass of the electron, the number of objective reals by the solid angle subtended at the point by the electron.

What is wanted at this stage is some simple hypothesis concerning the motion of objective reals and correlating it with the motion of electric points and electrons. From such a hypothesis the whole electromagnetic and gravitational laws might follow with the utmost simplicity. The complete concept involves the assumption of only one class of entities as forming the universe. Properties of "space" and of the physical phenomena "in space" become simply the properties of this single class of entities. In regard to the simplification of the preceding axioms, viz., of (I.–XVI.) Hp R, the ideal to be aimed at would be to deduce some or all of them from more general axioms which would also embrace the laws of physics. Thus these laws should not presuppose geometry, but create it.